Ideal quasi-normal convergence

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Ideal convergence

$\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an ideal if it is closed under subsets and finite unions, contains $\mathrm{Fin} = [\mathbb{N}]^{<\omega}$ and $\mathbb{N} \notin \mathcal{I}$.

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$ and let X be a set. Suppose that $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$, $(f_n) \subseteq \mathbb{R}^X$ and $f \in \mathbb{R}^X$.

- $x_n \xrightarrow{\mathcal{I}} x$ if $\{n : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$;
- $f_n \xrightarrow{\mathcal{I}} f$ (*$\mathcal{I}$-pointwise convergence*) if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for all $x \in X$;
- $f_n \xrightarrow{(\mathcal{I},\mathcal{J})\mathrm{QN}} f$ (*$(\mathcal{I}, \mathcal{J})$-quasi-normal convergence*) if there exists a sequence of positive reals $\varepsilon_n \xrightarrow{\mathcal{J}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for all $x \in X$.

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Theorem (Laczkovich and Recław; Debs and Saint Raymond; Filipów and Szuca)

Let $\mathcal{I}$ be a Borel ideal.

1. $Fin \otimes Fin \not\sqsubseteq \mathcal{I}$ *if and only if* $B^{\mathcal{I}}_{\alpha}(X) = B_{\alpha}(X)$ *for all* $1 \leq \alpha < \omega_1$ *and all perfectly normal spaces* X.
2. $Fin \otimes Fin \sqsubseteq \mathcal{I}$ *if and only if* $B^{\mathcal{I}}_{\alpha}(X) \supseteq B_{\alpha+1}(X)$ *for all* $1 \leq \alpha < \omega_1$ *and all perfectly normal spaces* X.

$\mathrm{Fin} \otimes \mathrm{Fin} \sqsubseteq \mathcal{I}$ means that we can find a partition $(A_n) \subseteq \mathcal{I}$ of $\mathbb{N}$ such that if $M \cap A_n$ is finite for each n, then $M \in \mathcal{I}$.

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Suppose that $\mathcal{I}$ and $\mathcal{J}$ are non-orthogonal and $\mathcal{I}$ is Borel.

1. c-type of $(\mathcal{I},\mathcal{J})$ is 1 if and only if $B_n^{(\mathcal{I},\mathcal{J})QN}(X) = B_n^{QN}(X)$ for all $1 \leq n < \omega$ and all perfectly normal spaces X.
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$$B_n^{QN}(\mathbb{R}) \subsetneq B_n(\mathbb{R}) \subsetneq B_{n+1}^{QN}(\mathbb{R})$$

In (2) and (3) the implications "$\implies$" hold for all $1 \leq \alpha < \omega_1$.

In fact it suffices to require that $\mathcal{I}$ is co-analytic.

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c-types

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Definition

1. c-type of $(\mathcal{I}, \mathcal{J})$ is 1 if $Fin \otimes Fin \not\sqsubseteq \mathcal{I} \sqcup (A_n)$ for any sequence $(A_n) \subseteq \mathcal{J}$.
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Example

1. c-type of $(\mathrm{Fin}, \mathrm{Fin} \otimes \mathrm{Fin})$ is 1.
2. c-type of $(\emptyset \otimes \mathrm{Fin}, \mathrm{Fin} \otimes \emptyset)$ is 2.
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ONE in his n-th turn picks $C_n \in \mathcal{I}$ and then
TWO responds with a finite set F_n with $F_n \cap C_n = \emptyset$.
ONE wins, if $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{I}$.
Otherwise TWO wins.

1. Martin's Theorem on Borel determinacy: the game is determined.
2. Laflamme: ONE has a winning strategy if and only if $\mathrm{Fin} \otimes \mathrm{Fin} \sqsubseteq \mathcal{I}$.
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- X is a QN-space if any sequence $(f_n) \subseteq \mathbb{R}^X$ of continuous functions converging to zero $(\mathrm{Fin}, \mathrm{Fin})$QN converges to zero.
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Theorem (Šupina)

If $Fin \otimes Fin \sqsubseteq \mathcal{I}$, then every topological space is an $\mathcal{I}$QN-space (and also $\mathcal{I}$wQN-space).

A sequence (U_n) of subsets of a topological space X is an $\mathcal{I}$-γ-cover if $U_n \neq X$ for all n and $\{n : x \notin U_n\} \in \mathcal{I}$ for all $x \in X$. $\mathcal{I}$-Γ is the family of all open $\mathcal{I}$-γ-covers.

The ideal version of Scheepers Conjecture does not hold if $\mathrm{Fin} \otimes \mathrm{Fin} \sqsubseteq \mathcal{I}$:

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Theorem (Šupina)

If $\mathfrak{p} = \mathfrak{c}$, then there is an ideal $\mathcal{I}$ such that $Fin \otimes Fin \not\subseteq \mathcal{I}$ and an $\mathcal{I}$QN-space which is not a QN-space.

However, this space is a wQN-space, so this result does not distinguish the notions of wQN-space and $\mathcal{I}$wQN-space.

$\mathcal{I}$ is tall if any infinite set contains an infinite subset from $\mathcal{I}$.

Theorem (Bukovský, Das and Šupina)

For non-tall ideals the notions of $\mathcal{I}$QN-space ($\mathcal{I}$wQN-space) and QN-space (wQN-space) coincide.

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$\mathcal{I}$wQN-spaces for tall ideals

Theorem

Let $\mathcal{I}$ be a tall ideal. Then any $\mathcal{I}$wQN-space of cardinality strictly less than $\mathrm{cov}^*(\mathcal{I})$ is a wQN-space.

$$\mathrm{cov}^*(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{X \in [\omega]^\omega} \exists_{A \in \mathcal{A}} |A \cap X| = \omega\right\}$$

- $\mathfrak{p} \leq \mathrm{cov}^*(\mathcal{I}) \leq \mathfrak{c}$ for any tall ideal;
- Keremedis: $\mathrm{cov}^*(\mathtt{nwd}) = \mathrm{cov}(\mathcal{M})$, where $\mathtt{nwd}$ consists of all nowhere dense subsets of $\mathbb{Q} \cap [0,1]$; $\mathtt{nwd}$ is $\mathrm{F}_{\sigma\delta}$;
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$\mathtt{non}(\mathcal{I}$QN-space) ($\mathtt{non}(\mathcal{I}$wQN-space)) denotes the minimal cardinality of a perfectly normal space which is not an $\mathcal{I}$QN-space ($\mathcal{I}$wQN-space).

Theorem (Filipów and Staniszewski; K; Šupina)

There are strictly combinatorial characterizations of $\mathtt{non}(\mathcal{I}$QN-space) and $\mathtt{non}(\mathcal{I}$wQN-space).

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$\mathfrak{b} \leq \mathtt{non}(\mathcal{I}\text{QN-space}) \leq \mathtt{non}(\mathcal{I}\text{wQN-space}) \leq \mathfrak{d}$ *for every* $\mathcal{I}$ *such that* $Fin \otimes Fin \not\sqsubseteq \mathcal{I}$.

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Nice ideals

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$\mathtt{non}(\mathcal{I}QN\text{-space}) = \mathtt{non}(\mathcal{I}wQN\text{-space}) = \mathfrak{b}$ *for all* $\mathtt{F}_\sigma$ *ideals and all analytic P-ideals.*

Theorem (Das and Chandra)

$\mathtt{add}(\mathcal{I}QN\text{-space}) \geq \mathfrak{b}$ *for every P-ideal.*

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Theorem

If $\mathfrak{b} < \mathfrak{b}_{\mathcal{J}}$ for some $\mathcal{J}$, then there are $\mathcal{I}$ with $Fin \otimes Fin \not\subseteq \mathcal{I}$ and an $\mathcal{I}$wQN-space which is not a wQN-space.

Proof.

Bukovský, Recław and Repický proved that $\mathrm{non}(\text{wQN-space}) = \mathfrak{b}$. We show $\mathrm{non}(\mathcal{I}\text{wQN-space}) \geq \mathfrak{b}_{\mathcal{J}}$. □

Consistently, the ideal version of Scheepers Conjecture does not hold even for some ideals with $\mathrm{Fin} \otimes \mathrm{Fin} \not\subseteq \mathcal{I}$:

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Thank you for your attention!

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