The Selective Strong Screenability Game

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Introduction

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Problem (Aleksandroff, 1947)

Is countable dimensional equivalent to weakly - infinite dimensional?

Selective Screenability Property

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Let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of a set S. The symbol $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

> *For each sequence $(\mathcal{U}_n : n < \omega)$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n < \omega)$ such that:*
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The property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ of a topological space is called *selective screenability* of the space. The property $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ is the Aleksandroff notion of *weakly - infinite dimensional* topological space.

Aleksandroff's Problem

Theorem (R. Pol, 1981)

There is a separable metric space which is weakly infinite dimensional, but not countable dimensional.

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4. $dim(A) \leq dim(B)$ whenever $A \subseteq B$.

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$$A_0,\ B_0,\ \cdots,\ A_\gamma,\ B_\gamma,\ \cdots \quad \gamma < \alpha$$

is won by TWO if $\bigcup\{B_\gamma : \gamma < \alpha\} \in \mathcal{B}$; otherwise, ONE wins.

Game Dimension

Define for a space S an ordinal $tp_c(\mathcal{O},\mathcal{O})(S)$ as

$$\min\{\alpha > 0 : \text{TWO has a winning strategy in the game } \mathsf{G}_c^{\alpha}(\mathcal{O},\mathcal{O}) \text{ on } S\}$$

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$$1 + dim_{G_c}(S) = tp_c(\mathcal{O}, \mathcal{O})(S)$$

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Note: The game dimension of Pol's counterexample to the Aleksandroff problem is $\omega + 1$.

Neighborhood Game Dimension

Theorem (Banach-Mazur, 1932)

Every separable metric space embeds isometrically into the topological group $(C[0,1], +, ||\cdot||_{max})$

For topological group $(H, *)$ with identity element e and a neighborhood U of e, $O(U) = \{x * U : x \in H\}$ is an open cover of H.

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Define *neighborhood game dimension* of subset S, $dim_{nbd}(S)$, by

$$1 + dim_{nbd}(S) = tp_c(\mathcal{O}_{nbd}, \mathcal{O}_S)(H)$$

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A collection $\mathcal{A}$ of subsets of a topological space (X, τ) is *discrete* if there is for each $x \in X$ a neighborhood U of x such that $|\{A \in \mathcal{A} : A \cap U \neq \emptyset\}| \leq 1$.

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Selective Strong Screenability and Paracompacteness

A topological space is *paracompact* if for each given open cover there is a locally finite open cover refining the given cover.

Theorem (Michael, 1953)

A regular space is paracompact if, and only if, it is strongly screenable.

Selective Strong Screenability and Paracompacteness

A topological space is *paracompact* if for each given open cover there is a locally finite open cover refining the given cover.

Theorem (Michael, 1953)

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Theorem (Nagami, 1955)

A normal, countably paracompact space is screenable if, and only if, it is strongly screenable.

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$\mathsf{G}_c^\alpha(\mathcal{A},\mathcal{B})$ vs. $\mathsf{G}_d^\alpha(\mathcal{A},\mathcal{B})$

Lemma

If TWO has a winning strategy in $\mathsf{G}_d^\alpha(\mathcal{A},\mathcal{B})$, *then TWO has a winning strategy in* $\mathsf{G}_c^\alpha(\mathcal{A},\mathcal{B})$.

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If ONE has a winning strategy in $\mathsf{G}_c^\alpha(\mathcal{A},\mathcal{B})$, then ONE has a winning strategy in $\mathsf{G}_d^\alpha(\mathcal{A},\mathcal{B})$.

The Game $\mathsf{G}^1_d(\mathcal{O},\mathcal{O})$

Lemma

For Lindelöf space X the following are equivalent:

1. *X is zero-dimensional.*
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The Game $\mathsf{G}_d^\omega(\mathcal{O},\mathcal{O})$

Theorem

Let X be a metrizable space and let Y be a subspace of X. If TWO has a winning strategy in the game $\mathsf{G}_d^\omega(\mathcal{O},\mathcal{O}_Y)$ on X, then Y is a subset of a union of countably many closed, strongly zero-dimensional subsets of X.

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Corollary

If X is a metrizable space, then the following are equivalent:

1. *TWO has a winning strategy in $G_d^{\omega}(\mathcal{O}, \mathcal{O})$.*
2. *TWO has a winning strategy in $G_d^{1}(\mathcal{O}, \mathcal{O})$.*

The game $\mathsf{G}_d^{\alpha}(\mathcal{O}, \mathcal{O})$ for $[0, 1]$

Theorem

TWO has a winning strategy in $\mathsf{G}_d^{\omega+1}(\mathcal{O}, \mathcal{O})$ *on the closed unit interval.*

The game $G_d^{\alpha}(\mathcal{O},\mathcal{O})$ for $[0,1]$

Theorem

TWO has a winning strategy in $G_d^{\omega+1}(\mathcal{O},\mathcal{O})$ on the closed unit interval.

Conjecture

For each positive integer n ONE has a winning strategy in $G_d^{\omega\cdot n}(\mathcal{O},\mathcal{O})$, and TWO has a winning strategy in $G_d^{\omega\cdot n+1}(\mathcal{O},\mathcal{O})$ on $[0,\ 1]^n$.

$\mathsf{G}_d^{\alpha}(\mathcal{O},\mathcal{O})$ and Possible Game Dimension

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Define for a subset S the *strong dimension* of S, $dim_{G_d}(S)$, by

$$\omega \cdot dim_{G_d}(S) + 1 = tp_d(\mathcal{O},\mathcal{O})(S)$$

Question

How does $dim_{G_d}(S)$ behave compared to $dim_{G_c}(S)$?

HAPPY 60^{th} BIRTHDAY!

Thank you!