Domain representable spaces and topological games

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Warszawa, 22^{nd} August 2017

Dcpo

We say that poset $(P, \sqsubseteq)$ is **directed complete (dcpo)** if every directed $D \subseteq P$ has a least upper bound.

Approximation relation

On dcpo $(P, \sqsubseteq)$ we specify a relation $\ll$ in the following way

$$x \ll y \quad \Leftrightarrow \quad \forall_{D \text{ directed}} \quad (y \sqsubseteq \sup D \quad \Rightarrow \quad \exists_{d \in D} \quad x \sqsubseteq d).$$

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Let

$$\Downarrow x = \{y \in P : y \ll x\}$$

for every $x \in P$.

Domain

A dcpo $(P, \sqsubseteq)$ is called **continuous** if

1. $\Downarrow x$ is directed,
2. $\sup \Downarrow x = x$,

for every $x \in P$. A continuous dcpo is called a **domain**.

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for every $x \in P$. A continuous dcpo is called a **domain**.

Scott topology

Let $(P, \sqsubseteq)$ be a domain. A set U is called **Scott open** if

1. $U = \bigcup_{x \in U} \uparrow x$,
2. $\sup D \in U \Rightarrow D \cap U \neq \emptyset$ for any directed set $D \subseteq P$.

A **Scott topology** on a domain $(P, \sqsubseteq)$ is the collection of all Scott open sets.

Domain representable space

We say that a space X is **domain representable** if there exist a domain $(P, \sqsubseteq)$ and a homeomorphism $h\colon X \to \max P$.

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Example

The partially ordered set:

$$P = \{[a,b] : a \leqslant b\}$$

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

The approximation relation on P:

$$[a,b] \ll [c,d] \Leftrightarrow [c,d] \subseteq (a,b)$$

The homeomorphism $h\colon \max P \to \mathbb{R}$:

$$h([x,x]) = x \quad \text{for } x \in X.$$

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Example

Let (X, d) be a complete metric space.

The partially ordered set:

$$\mathbb{B}X = X \times [0, \infty)$$

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leqslant r - s$$

The approximation relation on $\mathbb{B}X$:

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) < r - s$$

The homeomorphism $h\colon \max \mathbb{B}X \to X$:

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Theorem[Martin, 2003]

A metric space is a domain representable iff it is completely metrizable.

Domain representable space

We say that a triple $(Q, \ll, B)$ **represents** a space X and that X is **domain representable** if

(1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a base for $\tau(X)$,

(2) $\ll$ is a transitive, antisymmetric relation on Q,

(3) for all $p, q \in Q$, $p \ll q$ implies $B(p) \supseteq B(q)$,

(4) for all $x \in X$ if $p, p' \in \{q \in Q : x \in B(q)\}$, then there exists $r \in \{q \in Q : x \in B(q)\}$ satisfying $p, q \ll r$,

(5) if $D \subseteq Q$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$.

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Example

We consider a space

$$\sigma(\{0,1\}^{\omega_1}) = \{x \in \{0,1\}^{\omega_1} : |\alpha < \omega_1 : x(\alpha) = 1| \leqslant \omega\}$$

with the topology generated by the base

$$\mathcal{B} = \{pr_A^{-1}(x) : A \in [\omega_1]^{\leqslant\omega}, x \in \{0,1\}^A\},$$

where $pr_A : \sigma(\{0,1\}^{\omega_1}) \to \{0,1\}^A$ means projection for $A \in [\omega_1]^{\leqslant\omega}$.

Let $Q = \mathcal{B}$ and $B : Q \to Q$ be identity. We define a relation $\ll$ as follows

$$\mathrm{pr}_A^{-1}(x_A) \ll \mathrm{pr}_B^{-1}(x_B) \Leftrightarrow \mathrm{pr}_B^{-1}(x_B) \subseteq \mathrm{pr}_A^{-1}(x_A),$$

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Two players α and β alternately choose open nonempty sets with

β	U_0		U_1		
					$\dots$
α		V_0		V_1	

Player α wins this play if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise β wins.

Denoted this game by $BM(X)$.

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The *strong Choquet* game is defined as follows:

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A strategy and a winning strategy

A **strategy** for the player α in the game $BM(X)$ (or $Ch(X)$) is a rule for choosing what to play each round given the full information of moves up until that round.

A **winning strategy** for the player α is a strategy that produces a win for that player α in any game when playing according to that startegy.

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Theorem[Martin, 2003]

If a space X is domain representable, then the player α has a winning strategy in $Ch(X)$.

Theorem[Fleissner, Yengulalp, 2015]

If a space X is countably domain representable, then the player α has a winning strategy in $Ch(X)$.

Theorem[J.B., A. Kucharski]

If the player α has a winning strategy in $Ch(X)$, then X is countably domain representable.

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Main result

Theorem[J. B., A. Kucharski]

The player α has a winning strategy in the BM(X) iff X is countably π- domain representable.

Thank You for Your attention!