Compactifiable classes of compacta

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Our questions

We say that classes of topological spaces $\mathcal{C}$, $\mathcal{D}$ are *equivalent* if every member of $\mathcal{C}$ is homeomorphic to a member of $\mathcal{D}$ and vice versa. We write $\mathcal{C} \cong \mathcal{D}$.

Let $\mathcal{C}$ be a class of metrizable compacta.

Question

Can $\mathcal{C}$ be disjointly composed into one metrizable compactum such that the quotient space is also a metrizable compactum?

If $\mathcal{C}$ is a class of continua, then the question is equivalent to the following.

Question

Is there a metrizable compactum such that its set of connected components is equivalent to $\mathcal{C}$?

Compositions

Formally, a *composition* $\mathcal{A}$ consists of the following data:

$$\{A_b\}_{b \in B} \xrightarrow[\longrightarrow]{\{e_b\}_{b\in B}} A \xrightarrow{q_{\mathcal{A}}} B$$

- A, B are topological spaces,
- the maps $\{e_b\}_{b\in B}$ are embeddings such that $\{\mathrm{rng}(e_b)\}_{b\in B}$ is a decomposition of A,
- the map $q_{\mathcal{A}}$ defined by $q_{\mathcal{A}}^{-1}(b) = \mathrm{rng}(e_b)$ is continuous.

We write $\mathcal{A} = (A, e_b)_{b\in B}$ or $(A, A_b)_{b\in B}$ when $A_b \subseteq A$.

Rectangular compositions

Let A, B be topological spaces, let $F \subseteq A \times B$.

- We put $F^b := \{a \in A : (a, b) \in F\}$ for every $b \in B$.
- F induces the composition $\mathcal{A}_F := (F, e_b)_{b \in B}$ where $e_b \colon F^b \to F^b \times \{b\}$, so $q_{\mathcal{A}_F} = \pi_B \restriction F$.
- Compositions of this form are called *rectangular compositions*.

Every composition is equivalent to a rectangular composition.

- For $\mathcal{A} = (A, A_b)_{b \in B}$ it is enough to put $F := \{(a, q_{\mathcal{A}}(a)) : a \in A\} \subseteq A \times B$.
- We have $F^b = A_b$ for every $b \in B$.
- F is the graph of $q_{\mathcal{A}}$, which is closed.

Compactifiable and Polishable classes

Definition

A composition $\mathcal{A} = (A, A_b)_{b \in B}$ is called

- *compact* if A, B are metrizable compacta,
- *Polish* if A, B are Polish spaces.

Definition

A class of topological spaces $\mathcal{C}$ is compactifiable (or Polishable) if there exists a compact (or Polish) composition $(A, A_b)_{b \in B}$ such that $\{A_b : b \in B\} \cong \mathcal{C}$.

Equivalences

Theorem

The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

1. There is a compact (or Polish) composition $(A, A_b)_{b \in B}$ such that $\{A_b \colon b \in B\} \cong \mathcal{C}$.
2. There are metrizable compacta (or Polish spaces) A and B and a continuous map $q \colon A \to B$ such that $\{q^{-1}(b) \colon b \in B\} \cong \mathcal{C}$.
3. There are metrizable compacta (or Polish spaces) A and B and a closed (or G_δ) set $F \subseteq A \times B$ such that $\{F^b \colon b \in B\} \cong \mathcal{C}$.
4. There is a closed (or G_δ) set $F \subseteq [0,1]^\omega \times 2^\omega$ (or $\times\, \omega^\omega$) such that $\{F^b : b \in 2^\omega$ (or ω^ω)$\} \cong \mathcal{C}$.

Observations

- Compactifiable and Polishable classes are stable under countable unions – consider the one-point compactification of

$$\sum_{i\in I} q_i \colon \sum_{i\in I} A_i \to \sum_{i\in I} B_i.$$

- Hence, every countable family of metrizable compacta (or Polish spaces) is compactifiable (or Polishable).
- On the other hand, a cardinal argument gives that there are many classes of metrizable compacta that are not Polishable.
  - There are $\mathfrak{c}$-many G_δ subsets of $[0,1]^\omega \times \omega^\omega$.
  - There are $\mathfrak{c}$-many non-homeomorphic metrizable compacta, and so $2^{\mathfrak{c}}$-many non-equivalent classes.

Hyperspaces

For a topological space X we shall consider the hyperspaces of all subsets $\mathcal{P}(X)$, all closed subsets $\mathcal{C}l(X)$, all compact subsets $\mathcal{K}(X)$, and all subcontinua $\mathcal{C}(X)$ endowed with the Vietoris topology.

Recall

- The Vietoris topology is generated by the sets

  $U^- = \{A \subseteq X : A \cap U \neq \emptyset\}$ and $U^+ = \{A \subseteq X : A \subseteq U\}$

  for open $U \subseteq X$.
- $\mathcal{K}(X)$ is metrizable by the Hausdorff metric

  $$d_H(A, B) = \max(\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)).$$

  if X is metrizable by a metric d.
- $\mathcal{K}(X)$ is compact (or Polish) if X is compact (or Polish).
- $\mathcal{C}(X)$ is closed in $\mathcal{K}(X)$ if X is Hausdorff.
- $\mathcal{R}_\in = \{(x, A) : x \in A \in \mathcal{C}l(X)\}$ is closed if X is regular.

Compositions and hyperspaces

Definition

A composition $\mathcal{A} = (A, A_b)_{b \in B}$ is *strong* if $q_{\mathcal{A}}$ is closed and open and $|B \setminus \operatorname{rng}(q_{\mathcal{A}})| \leq 1$. We also define *strongly compactifiable* and *strongly Polishable* classes.

Construction

$$\mathcal{A}_{\mathcal{F}}\,(\pi_{\mathcal{F}} \colon \mathcal{R}_{\in} \cap (X \times \mathcal{F}) \to \mathcal{F})$$

$$X,\ \mathcal{F} \subseteq \mathcal{P}(X) \qquad\qquad \mathcal{A}\,(q \colon A \to B)$$

$$A,\ \mathcal{F}_{\mathcal{A}} := \{q^{-1}(b)\}_{b \in B} \subseteq \mathcal{P}(A)$$

- If X is metrizable and $\mathcal{F} \subseteq \mathcal{K}(X)$, then $\mathcal{A}_{\mathcal{F}}$ is strong.
- A composition $\mathcal{A}$ is strong if and only if $\mathcal{F}_{\mathcal{A}} \cong B$ via ${q_{\mathcal{A}}^{-1}}^{*}$.

Compositions and hyperspaces

Theorem

The following conditions are equivalent for a class of compacta $\mathcal{C}$.

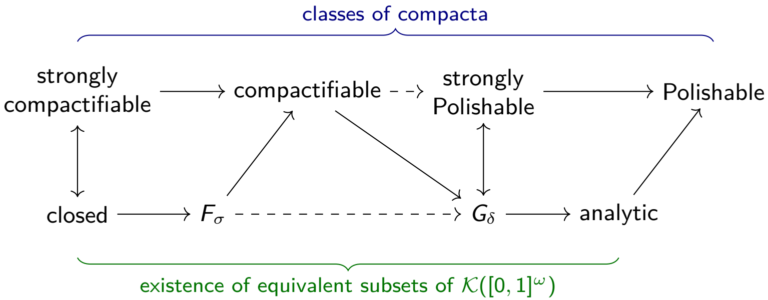
1. $\mathcal{C}$ is strongly compactifiable (or strongly Polishable).
2. There is a metrizable compactum (or a Polish space) A and a closed (or G_δ) family $\mathcal{F} \subseteq \mathcal{K}(A)$ such that $\mathcal{F} \cong \mathcal{C}$.
3. There is a closed (or G_δ) family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

Proposition

Let $\mathcal{A} = (A, A_b)_{b \in B}$ be a Polish composition of compacta.

- If $q_{\mathcal{A}}$ is closed, then $\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{K}(A)$ is G_δ.
- Every compactifiable class is strongly Polishable class.

Implications between the classes considered



Induced classes

Let $\mathcal{C}$ be a class of metrizable compacta. We consider the following classes of metrizable compacta induced by members of $\mathcal{C}$:

- the class of all subspaces $\mathcal{C}^{\downarrow}$
- the class of all superspaces $\mathcal{C}^{\uparrow}$
- the class of all homeomorphic copies $\mathcal{C}^{\cong}$
- the class of all continuous images $\mathcal{C}^{\twoheadrightarrow}$

Proposition

- $\mathcal{C}$ compactifiable $\implies$ $\mathcal{C}^{\downarrow}$ strongly compactifiable
- $\mathcal{C}$ Polishable $\implies$ $\mathcal{C}^{\downarrow}$ strongly Polishable
- $\mathcal{C}$ strongly compactifiable $\implies$ $\mathcal{C}^{\uparrow}$ strongly compactifiable
- $\mathcal{C}$ strongly Polishable $\implies$ $\mathcal{C}^{\uparrow}$ Polishable
- $\mathcal{C}$ strongly Polishable $\implies$ $\mathcal{C}^{\twoheadrightarrow}$ Polishable
- $\mathcal{C}$ strongly Polishable, X Polish $\implies$ $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ analytic

Induced classes

Examples

- Every hereditary class of metrizable compacta with a universal element is strongly compactifiable – all compacta, all continua, continua with dimension at most n, chainable continua, tree-like continua, dendrites.
- Every class of metrizable compacta closed under continuous images with a common model is Polishable – Peano continua, weakly chainable continua.
- The class of all uncountable compacta is strongly compactifiable.
- Classes coanalytically complete in $\mathcal{K}([0,1]^\omega)$ are not strongly Polishable – hereditarily decomposable continua, dendroids, λ-dendroids, arcwise connected continua, uniquely arcwise connected continua, hereditarily locally connected continua.

Compositions and inverse limits

Construction

Let $T \subseteq \omega^{<\omega}$ be a tree and let $\mathcal{D}$ be an inverse system of a shape T, i.e. for every $t \in T$ let X_t be a topological space and for every $t\,\hat{}\,k \in T$ let $f_{t,k}\colon X_{t\hat{}k} \to X_t$ be a continuous map.

- Let T_n denote the n-th level of T and let $[T] \subseteq \omega^\omega$ denote the space of all infinite branches of T.
- For every $\alpha \in [T]$ let $\mathcal{D}_\alpha$ be the inverse sequence
$$\mathcal{D} \restriction \alpha = (X_{\alpha \restriction n},\ f_{\alpha \restriction n,\, \alpha(n)})_{n\in\omega}.$$
- Let $\mathcal{D}^\oplus$ be the inverse sequence
$$\left(\sum_{t\in T_n} X_t,\ \sum_{t\in T_n}(\nabla_{t\hat{}k\in T}\, f_{t,k})\right)_{n\in\omega}.$$
- For every $\alpha \in [T]$ let $\eta_\alpha\colon \mathcal{D}_\alpha \hookrightarrow \mathcal{D}^\oplus$ be the transformation
$$(X_{\alpha\restriction n} \hookrightarrow \sum_{t\in T_n} X_t)_{n\in\omega}.$$

We obtain the composition $\mathcal{A}_\mathcal{D} := (\varprojlim \mathcal{D}^\oplus, \varprojlim \eta_\alpha)_{\alpha\in[T]}$ of the family $\{\varprojlim \mathcal{D}_\alpha\}_{\alpha\in[T]}$.

Compositions and inverse limits

Construction

...the composition $\mathcal{A}_{\mathcal{D}} = (\varprojlim \mathcal{D}^{\oplus}, \varprojlim \eta_\alpha)_{\alpha \in [T]}$ of $\{\varprojlim \mathcal{D}_\alpha\}_{\alpha \in [T]}$.

Observation

- $[T]$ is a closed subset of the Polish space ω^ω, and it is compact if T is finitely splitting.
- $\mathcal{A}_{\mathcal{D}}$ is a Polish composition if all spaces X_t are Polish.
- $\mathcal{A}_{\mathcal{D}}$ is a compact composition if all spaces X_t are metrizable compacta and the tree T is finitely splitting.

Compositions and inverse limits

A topological space is

- *(weakly) $\mathcal{P}$-like* if it is an inverse limit of a sequence with spaces from $\mathcal{P}$ and continuous bonding maps that are (not necessarily) onto.
- *$\mathcal{F}$-like* if it is an inverse limit of a sequence with maps from $\mathcal{F}$.

Proposition

Let $\mathcal{F}$ be a countable family of continuous maps and let $\mathcal{C}$ be the class of all $\mathcal{F}$-like spaces.

1. There exists an inverse system $\mathcal{D}$ of a shape $T \subseteq \omega^\omega$ such that $\{\mathcal{D}_\alpha\}_{\alpha \in [T]} \cong \mathcal{C}$, so if the (co)domains of $\mathcal{F}$ are Polish, then $\mathcal{C}$ is Polishable.
2. If $\mathrm{id}_X \in \mathcal{F}$ for every X that is a codomain of infinitely many maps from $\mathcal{F}$, then we may get a finitely splitting T. So if the (co)domains of $\mathcal{F}$ are metrizable compacta, then $\mathcal{C}$ is compactifiable.

Compositions and inverse limits

Proposition

Let $\mathcal{G}$ be a family of continuous maps such that the (co)domains of $\mathcal{G}$ form a countable set of metrizable compacta. There exists countable $\mathcal{F} \subseteq \mathcal{G}$ such that every space is $\mathcal{F}$-like iff it is $\mathcal{G}$-like.

Theorem

For every countable family of metrizable compacta $\mathcal{P}$ the classes of all (weakly) $\mathcal{P}$-like spaces are compactifiable.

Example

- The classes of all arc-like and all circle-like continua are compactifiable.
- The whole construction stems from the construction of a universal arc-like continuum [12.22, Nadler].
- There is no universal circle-like continuum.

Problems

Questions

- Is there a compactifiable class that is not strongly compactifiable?
- Are strongly compactifiable classes closed under countable unions?
- Is there a Polishable class that is not compactifiable?
- Is the class of all Peano continua compactifiable?

Thank you for your attention.