Selections Principles and Combinatorics of Open Covers,

Frontiers of Selection Principles, Warsaw

Lev Bukovský

Institute of Mathematics,
P.J. Šafárik University, Košice

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(7 October 1926 – 18 September 2015)

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Anyway, as far as I know, nobody found and proved those results yet, with exception of some special cases.
Moreover, those special cases were considered as important.

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Any infinite subset of a γ-cover is a γ-cover.

An open φ-cover $\mathcal{U}$ is **shrinkable**, if there exists an open φ-cover $\mathcal{V}$ such that

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If $\Phi, \Psi \subseteq \mathcal{P}(Y)$ are sets of subsets of Y, then $\mathsf{S}_1(\Phi, \Psi)$ means the following: For any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of Φ, for every $n \in \omega$ there exists an $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\} \in \Psi$.

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We shall consider $\mathsf{S}_1(\Phi, \Psi)$ and $\mathsf{S}_1(\Phi^{sh}, \Psi)$ for $\Phi, \Psi = \mathcal{O}, \Lambda, \Omega, \Gamma$.

No (infinite Hausdorff) topological space is an $\mathrm{S}_1(\mathcal{O}, \Lambda)$-space.

So, neither an $\mathsf{S}_1(\mathcal{O}, \Gamma)$-space nor an $\mathsf{S}_1(\mathcal{O}, \Gamma)$-space.

No infinite regular topological space is an $\mathrm{S}_1(\mathcal{O}^{sh}, \Lambda)$-space.

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One can easily prove (see also [JMSS]) that

$$\mathrm{S}_1(\Omega_{cnt}, \mathcal{O}) = \mathrm{S}_1(\Omega_{cnt}, \Lambda), \quad \mathrm{S}_1(\Gamma, \mathcal{O}) = \mathrm{S}_1(\Gamma, \Lambda).$$

$$\mathrm{S}_1((\Omega^{sh})_{cnt}, \mathcal{O}) = \mathrm{S}_1((\Omega^{sh})_{cnt}, \Lambda), \quad \mathrm{S}_1(\Gamma^{sh}, \mathcal{O}) = \mathrm{S}_1(\Gamma^{sh}, \Lambda).$$

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$$\mathrm{C}_p(X) \subseteq {}^X\mathbb{R}, \quad \mathrm{USC}(X) \subseteq {}^X\mathbb{R}, \quad \mathrm{C}_p(X)^+, \quad \mathrm{USC}(X)^+$$

$f \in {}^X\mathbb{R}$ is upper semicontinuous if for any real a the set $\{x \in X : f(x) < a\}$ is open.

$\min\{f_0, \ldots, f_k\}$ is the function h defined as

$$h(x) = \min\{f_0(x), \ldots, f_k(x)\} \text{ for } x \in X.$$

$F \subseteq {}^X\mathbb{R}$, $\varepsilon > 0$. We define

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If f is continuous or positive upper semicontinuous then each U_f^{ε} is an open set.

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$(\mathcal{O})$ $0 \in \overline{\{f(x) : f \in F\}}$ *for every* $x \in X$.

(Ω) $\mathbf{0} \in \overline{F}$ *in the topology of* ${}^X\mathbb{R}$.

(Γ) *there exists a countable* $H = \{h_n : n \in \omega\} \subseteq F$ *such that* $h_n \to \mathbf{0}$ *on* X.

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Evidently

$$(\Gamma) \to (\Omega) \to (\mathcal{O}).$$

Theorem

Let F be a family of positive upper semicontinuous functions.

a) *For $\Phi = \mathcal{O}, \Omega$ the family F possesses the property (Φ) if and only if either there exists a subsequence of F uniformly converging to* **0** *or there exists a $\delta > 0$ such that for every positive $\varepsilon < \delta$ the family $\mathcal{U}^\varepsilon(F)$ is an open φ-cover of X.*

b) *The family F possesses the property (Γ) if and only if there exists a countable infinite family $H \subseteq F$ such that either the family H uniformly converges to* **0** *or there exists a $\delta > 0$ such that for every positive $\varepsilon < \delta$ the family $\mathcal{U}^\varepsilon(H)$ is an open γ-cover of X.*

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If $\delta = 0$, then for every $n \in \omega$ we have $X \in \mathcal{U}^{2^{-n}}(F)$. For every n there exists a function $f_n \in F$ such that $X = U_{f_n}^{2^{-n}}$. Then $f_n \rightrightarrows \mathbf{0}$.
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So assume $\delta > 0$.
Let $\varepsilon < \delta$. Then for any finite set $\{x_0, \dots, x_k\} \subseteq X$ there exists an $f \in F$ such that $f \in N^{\varepsilon}_{x_0,\dots,x_k}$. Then $x_0, \dots, x_k \in U^{\varepsilon}_f$. Hence, the family $\mathcal{U}^{\varepsilon}(F)$ is an ω-cover.

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Let $\varepsilon < \delta$. Then for any finite set $\{x_0, \ldots, x_k\} \subseteq X$ there exists an $f \in F$ such that $f \in N^{\varepsilon}_{x_0,\ldots,x_k}$. Then $x_0, \ldots, x_k \in U^{\varepsilon}_f$. Hence, the family $\mathcal{U}^{\varepsilon}(F)$ is an ω-cover.
The proof of opposite implication is similar. If there exists a sequence $\langle f_n : n \in \omega \rangle \subseteq F$ such that $f_n \rightrightarrows \mathbf{0}$, then F trivially possesses the property (Ω). Otherwise for any positive $\varepsilon < \delta$ and $x_0, \ldots, x_k \in X$ there exists $f \in F$ such that $x_0, \ldots, x_k \in U^{\varepsilon}_f$. Then $f \in N^{\varepsilon}_{x_0,\ldots,x_k}$. Consequently $\mathbf{0} \in \overline{F}$. □

$\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}({}^X(0,\infty))$ are sets of sets of positive real functions.

$\mathsf{S}_1(\mathcal{F}, \mathcal{G})$ is a selection principle: for any sequence $\langle F_n : n \in \omega \rangle$ of sets of $\mathcal{F}$, for each $n \in \omega$ there exists an $f_n \in F_n$ such that the family $\{f_n : n \in \omega\}$ belongs to $\mathcal{G}$.

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Let Φ and Ψ be one of the symbols $\mathcal{O}, \Omega, \Gamma$. For $G \subseteq {}^X(0,\infty)$ we denote

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If $\mathsf{S}_1(\{\Phi\}(G), \{\Psi\}(G))$ holds true, we say that G satisfies the Selection Principle $\mathsf{S}_1(\{\Phi\}, \{\Psi\})$.

Similarly, identifying the countable sets of functions with sequences, if $\mathsf{S}_1((\{\Phi\}(G))_{cnt}, \{(\Psi\}(G))_{cnt})$ holds true, we say that G satisfies the Sequence Selection Principle $\mathsf{S}_1(\{\Phi\}_{cnt}, \{\Psi\}_{cnt})$

If $\Psi = \Omega, \Gamma$ and X is a topological space then the family $\mathrm{USC}(X)^+$ does not possess $\mathsf{S}_1(\{\mathcal{O}\}, \{\Psi\})$.

If $\Psi = \Omega, \Gamma$ and X is a Tychonoff topological space then the family $\mathrm{C}_p(X)$ does not possess $\mathsf{S}_1(\{\mathcal{O}\}, \{\Psi\})$.

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Following [Sch1] we obtain

If $F \subseteq {}^X(0, \infty)$ is closed under operation of taking minimum of two functions, then the family F satisfies $\mathsf{S}_1(\{\mathcal{O}\}, \{\mathcal{O}\})$ if and only if F satisfies $\mathsf{S}_1(\{\Omega\}, \{\mathcal{O}\})$.

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Assume that Φ *is one of the symbols* Ω, Γ *and* Ψ *is some of the symbols* $\mathcal{O}$, Ω, Γ. *Then then for any couple* $\Phi\Psi$ *different from* $\Omega\mathcal{O}$ *a topological space* X *is an* $\mathrm{S}_1(\Phi, \Psi)$*-space if and only if the family* $\mathrm{USC}(X)^+$ *satisfies the Selection Principle* $S_1(\{\Phi\}, \{\Psi\})$.

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Theorem

Assume that Φ is one of the symbols Ω, Γ and Ψ is some of the symbols $\mathcal{O}$, Ω, Γ. Then then for any couple $\Phi\Psi$ different from $\Omega\mathcal{O}$ a normal topological space X is an $\mathrm{S}_1(\Phi^{sh}, \Psi)$-space if and only if $\mathrm{C}_p(X)^+$ satisfies the Selection Principle $S_1(\{\Phi\}, \{\Psi\})$.

Lemma

Let φ be one of the symbols λ, ω or γ. Assume that $\langle \varepsilon_n : n \in \omega \rangle$ is a sequence of positive reals converging to 0, $f_n \in {}^X\mathbb{R}$ for any $n \in \omega$. If $\{U^{\varepsilon_n}_{f_n} : n \in \omega\}$ is a φ-cover then either there exists an increasing sequence $\langle n_k : k \in \omega \rangle$ of integers such that $f_{n_k} \rightrightarrows \mathbf{0}$ or there exists a $\delta > 0$ such that the family $\mathcal{U}^{\varepsilon}(\{f_n : n \in \omega\})$ is a φ-cover for every positive $\varepsilon < \delta$.

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Proof: Let $\delta = \inf\{\varepsilon > 0 : (\exists m)\, X = U^{\varepsilon}_{f_m} \vee \varepsilon = 1\}$.
If $\delta = 0$ then as above we can find a subsequence of $\{f_n : n \in \omega\}$ uniformly converging to $\mathbf{0}$.

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If $\delta = 0$ then as above we can find a subsequence of $\{f_n : n \in \omega\}$ uniformly converging to $\mathbf{0}$.
So assume that $\delta > 0$. Then for any positive $\varepsilon < \delta$ there exists an n_0 such that $\varepsilon_n < \varepsilon$ for each $n \geq n_0$. Then $U^{\varepsilon_n}_{f_n} \subseteq U^{\varepsilon}_{f_n}$ for $n \geq n_0$. Omitting finitely many elements of the φ-cover $\{U^{\varepsilon_n}_{f_n} : n \in \omega\}$ you obtain a φ-cover $\{U^{\varepsilon_n}_{f_n} : n \geq n_0\}$. Hence both $\{U^{\varepsilon}_{f_n} : n \geq n_0\}$ and $\mathcal{U}^\varepsilon(\{f_n : n \in \omega\}) = \{U^{\varepsilon}_{f_n} : n \in \omega\}$ are φ-covers as well. □

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Assume that the family $\mathrm{USC}(X)^+$ satisfies the Selection Principle $\mathsf{S}_1(\{\Phi\},\{\Psi\})$. Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of open φ-covers. For any $U \in \mathcal{U}_n$, $n \in \omega$ set

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For $\varepsilon \leq 1$, $n \in \omega$, $U \in \mathcal{U}_n$, $2^{-m+1} < \varepsilon$: $\quad U^{\varepsilon}_{f_U + 2^{-m}} = U$.
Otherwise $U^{\varepsilon}_{f_U + 2^{-m}} = \emptyset$.

Hence we can conclude that $\mathcal{U}^{\varepsilon}(F_n)$ is a φ-cover.

Thus, every set F_n possesses the property (Φ). By the Selection Principle $\mathsf{S}_1(\{\Phi\}(USC^+(X)), \{\Psi\}(USC(X)^+))$ for each $n \in \omega$ there exists $f_n \in F_n$ such that the set $\{f_n : n \in \omega\}$ satisfies the property (Ψ). For every n there exist a set $U_n \in \mathcal{U}_n$ and $m_n \in \omega$ such that $f_n = f_{U_n} + 2^{-m_n}$.

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Since for all but finitely many n we have $2^{-m_n+1} < \varepsilon$, the set $\{U_n : n \in \omega\}$ and $\mathcal{U}^\varepsilon(\{f_n : n \in \omega\})$ differs in finitely many elements. Thus, for $\psi = \omega, \gamma$ the set $\{U_n : n \in \omega\}$ is a ψ-cover.

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Let $\varphi = o$

If $\mathcal{U}^\varepsilon(\{f_n : n \in \omega\}$ is a cover for all sufficiently small ε, then it is also a large cover for all sufficiently small ε.

Continue as above.

Let X be an $\mathsf{S}_1(\Phi, \Psi)$-space. Assume that $\langle F_n : n \in \omega \rangle$ is a sequence of sets of positive real upper semicontinuous functions and each set F_n possesses the property (Φ). The elements of any family $\mathcal{U}^\varepsilon(F_n)$ are open subsets of X.
If $\Phi = \mathcal{O}$ or $\Phi = \Omega$, then for every n either there exists a subsequence of F_n uniformly converging to $\mathbf{0}$ or there exists a $\delta_n > 0$ such that for every $\varepsilon < \delta_n$ the family $\mathcal{U}^\varepsilon(F_n)$ is a φ-cover.
Let A be the set of those $n \in \omega$ for which there exists a $\delta_n > 0$ such that for every $\varepsilon < \delta_n$ the family $\mathcal{U}^\varepsilon(F_n)$ is a φ-cover. We set

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If $\Phi = \Gamma$ then for each n there exists a countable set $H_n \subseteq F_n$ such that either H_n uniformly converges to $\mathbf{0}$ or there exists a $\delta_n > 0$ such that for every $\varepsilon < \delta_n$ the family $\mathcal{U}^\varepsilon(H_n)$ is a φ-cover. Again, let A be the set of those $n \in \omega$ for which there exists $\delta_n > 0$ with suitable property.

If A is finite, one can easily find a sequence $f_n \in F_n$ uniformly converging to $\mathbf{0}$. The family $\{f_n : n \in \omega\}$ possesses the property (Ψ).

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So let A be infinite. If $\Phi \neq \Gamma$, apply $\mathsf{S}_1(\Phi, \Psi)$ to the sequence of covers $\{\mathcal{U}^{\varepsilon_n}(F_n) : n \in A\}$. You obtain sets $U_n \in \mathcal{U}^{\varepsilon_n}(F_n)$, $n \in A$ such that the family $\{U_n : n \in A\}$ is a ψ-cover.

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In both cases, for every $n \in A$ there exists a function $f_n \in F_n$ such that $U_n = U_{f_n}^{\varepsilon_n}$. By Lemma 4 either there exists a subsequence uniformly converging to $\mathbf{0}$ or there exists a $\delta > 0$ such that for every $\varepsilon < \delta$ the family $\{U_{f_n}^{\varepsilon} : n \in \omega\}$ is a λ-, ω- or γ-cover for each positive $\varepsilon < \delta$, respectively. Therefore the family $\{f_n : n \in A\}$ possesses the property (Ψ).

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For $n \notin A$ take $f_n \in F_n$ arbitrary. Then the family $\{f_n : n \in \omega\}$ possesses the property (Ψ) as well.

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From right to left:
Since for every $n \in \omega$ the φ-cover $\mathcal{U}_n$ is shrinkable, there exists a φ-cover $\mathcal{V}_n$ with suitable property. For $V \in \mathcal{V}_n$ let $U_V \in \mathcal{U}_n$ be such that $\overline{V} \subseteq U_V$. Then $\{U_V : V \in \mathcal{V}_n\}$ is a φ-cover. Since X is a normal topological space, for every $V \in \mathcal{V}_n$ there exist a continuous real function $f_V : X \longrightarrow \langle 0, 1\rangle$ such that

$$f_V(x) = \begin{cases} 0 & \text{if } x \in \overline{V}, \\ 1 & \text{if } x \in X \setminus U_V. \end{cases}$$

As above, we set

$$F_n = \{f_V + 2^{-m} : V \in \mathcal{V}_n \wedge m \geq n\}.$$

Theorem

Let Ψ *be one of the symbols* $\mathcal{O}$, Ω, Γ.

1) *A topological space* X *is an* $\mathrm{S}_1(\Omega_{cnt}, \Psi)$*-space if and only if the family of all sequences of positive real upper semicontinuous functions on* X *satisfies* $S_1(\{\Omega\}, \{\Psi\})$.
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$$V_{U,n} = \{x \in X : f(x) > 2^{-n}\}.$$

Then $\overline{V}_{U,n} \subseteq V_{U,n+1} \subseteq U$ and $\bigcup_n V_{U,n} = U$. The cover

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If $\mathcal{U}$ is an ω-cover, then $\mathcal{V}$ is an ω-cover. □

Theorem

If X is perfectly normal, then for $\Phi = \Lambda, \Omega$ and $\Psi = \Lambda, \Omega, \Gamma$ we have

$$\mathrm{S}_1(\Phi^{sh}, \Psi) = \mathrm{S}_1(\Phi, \Psi).$$

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Corollary

Assume that X is a perfectly normal topological space, $\Phi = \Lambda, \Omega$ and $\Psi = \Lambda, \Omega, \Gamma$. If $\mathrm{C}_p(X)$ satisfies the Selection Principle $S_1(\{\Phi\}, \{\Psi\})$ then also the family $\mathrm{USC}(X)^+$ satisfies the Selection Principle $S_1(\{\Phi\}, \{\Psi\})$.

Theorem

Let X be a Tychonoff topological space, Ψ being one of the symbols $\mathcal{O}, \Omega, \Gamma$. Then the following are equivalent:

(i) *X is an $\mathrm{S}_1(\Omega, \Psi)$-space.*

(ii) *$\mathrm{USC}(X)^+$ satisfies the Selection Principle $\mathrm{S}_1(\{\Omega\}, \{\Psi\})$.*

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Proof: The implication $(\mathrm{i}) \to (\mathrm{ii})$ is a case of the first mainTheorem. $(\mathrm{ii}) \to (\mathrm{iii})$ is trivial. We show $(\mathrm{iii}) \to (\mathrm{i})$.
Since X is Tychonoff, for each open U and finitely many $x_0, \ldots, x_k \in U$ there exists a continuous function $f_{U,x_0,\ldots,x_k} : X \longrightarrow [0,1]$ such that $f_{U,x_0,\ldots,x_k}(x) = 1$ for $x \in X \setminus U$ and $f_{U,x_0,\ldots,x_k}(x_i) = 0$ for $i = 0, \ldots, k$.

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Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of ω-covers. Set

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We show it, e.g., for $\psi = \gamma$.
Let $x \in X$, $\varepsilon \leq 1$. Then there exists n_0 such that for $n \geq n_0$

$$f_n(x) = f_{U_n, x_0, \dots, x_{k_n}} + 2^{-m_n}(x) < \varepsilon.$$

Since $f_{U_n, x_0, \dots, x_{k_n}}(x) < 1$, we have $x \in U_n$. □

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$$
\begin{array}{ccccc}
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\downarrow & & \downarrow & & \downarrow \\
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\end{array}
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W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki proved that the properties in the first two rows are mutually non-equivalent.

M. Scheepers proving that every $\mathsf{S}_1(\Gamma,\Gamma)$-space is a wQN-space, i.e. that $\mathrm{C}_p(X)$ satisfies the Sequence Selection Principle $\mathsf{S}_1(\{\Gamma\},\{\Gamma\})$, conjectured that the opposite implication is true for normal topological spaces as well, i.e., if $\mathrm{C}_p(X)$ satisfies the Sequence Selection Principle $\mathsf{S}_1(\{\Gamma\},\{\Gamma\})$ then X is an $\mathsf{S}_1(\Gamma,\Gamma)$-space.

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For normal topological space neither the consistency of the negation of Scheepers Conjecture is known.

Some special cases of Main Theorems were already known.

The first Theorem for $\Phi = \Psi = \Gamma$ was proved by the author and J. Haleš [BH].

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X has countable strong tightness if for every $x \in X$ and every sequence $\langle A_n : n \in \omega \rangle$ of subsets of X such that $x \in \overline{A_n}$ for each $n \in \omega$, there exists a sequence $x_n \in A_n$, $n \in \omega$ such that $x \in \overline{\{x_n : n \in \omega\}}$.

For Tychonoff space the case $\Psi = \Omega$ of the last Theorem is a result by M. Sakai [Sa1]

$$\mathrm{S}_1(\Omega(X), \Omega(X)) \equiv \mathrm{C}_p(X) \textit{ has countable strong tightness.}$$

Thanks
for your attention!

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