Winter School in Abstract Analysis
sectionSet Theory & Topology

27th Jan – 3th Feb 2018

TUTORIAL SPEAKERS

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REGISTRATION DEADLINE
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Mathias–Příkrý forcing and covering properties of filters

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An alternative proof of a theorem of Pawlikowski

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D. Chodounský, D. Repovš, L. Zdomskyy, *Mathias forcing and combinatorial covering properties of filters*, Journal of Symbolic Logic 80 (2015), no. 4, 1398–1410.

D. Chodounský, O. Guzmán, M. Hrušák, *Mathias–Prikry and Laver type forcing; Summable ideals, coideals, and* $+$*-selective filters*, Archive for Mathematical Logic 55 (2016), no. 3-4, 493–504.

Filters and $\uparrow$-covers

For $a \subset \omega$ denote $\uparrow a = \{x \subset \omega \colon a \subset x\}$.

Fact

- *$\uparrow a$ is compact*
- *a is finite $\Rightarrow \uparrow a$ is open*

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Definition

Let $\mathcal{F} \subset 2^\omega$ be a filter. O is an *$\uparrow$-cover* of $\mathcal{F}$ if O is a cover of $\mathcal{F}$ consisting of sets of the form $\uparrow a$, $a \in [\omega]^{<\omega}$. We write $O \in \uparrow \mathcal{O}$.

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- $O \in \uparrow\mathcal{O} \quad \Rightarrow \quad (\forall n \in \omega)\ O_{n\leq} = (O \setminus \{\uparrow a : a \cap n \neq \emptyset\}) \in \uparrow\mathcal{O}$

Covers of filters

Proposition

Let $\mathcal{F}$ be a filter.

- $\mathbf{S}_{\text{fin}}(\mathcal{O},\mathcal{O})$ *iff* $\mathbf{S}_{\text{fin}}(\uparrow\mathcal{O},\mathcal{O})$
- $\mathbf{S}_{\text{fin}}(\mathcal{O},\Gamma)$ *iff* $\mathbf{S}_{\text{fin}}(\uparrow\mathcal{O},\Gamma)$

Lemma

Let $\mathcal{F}$ be a filter on ω, O a cover of $\mathcal{F}$ (consisting of open subsets of 2^ω). There is an $\uparrow$-cover U of $\mathcal{F}$, such that $\mathcal{F} \subset \bigcup U \subset \bigcup O$.

Properties of $\mathbb{M}(\mathcal{F})$

Let $\mathcal{F}$ be a filter on ω. The following are equivalent:

1	$\mathcal{F}$ is Hurewicz	$\mathcal{F}$ is Menger
2	no continuous image of $\mathcal{F}$ is unbounded in ω^ω	no continuous image of $\mathcal{F}$ is dominating in ω^ω
3	$\mathbf{S}_{\mathrm{fin}}(\uparrow\mathcal{O}, \Gamma)$	$\mathbf{S}_{\mathrm{fin}}(\uparrow\mathcal{O}, \mathcal{O})$

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3	$\mathbf{S}_{\mathrm{fin}}(\uparrow\mathcal{O}, \Gamma)$	$\mathbf{S}_{\mathrm{fin}}(\uparrow\mathcal{O}, \mathcal{O})$
4	$\mathbb{M}(\mathcal{F})$ preserves unbounded sets in $\langle\omega^\omega, <^*\rangle$ as unbounded	$\mathbb{M}(\mathcal{F})$ preserves dominating sets in $\langle\omega^\omega, <^*\rangle$ as unbounded
5	$\mathbb{M}(\mathcal{F})$ is almost ω^ω bounding	$\mathbb{M}(\mathcal{F})$ does not add dominating reals

Definition (Mathias–Příkrý forcing)

Let $\mathcal{F}$ be a filter on ω.

$\mathbb{M}(\mathcal{F}) = \{\langle a, F\rangle \colon a \in [\omega]^{<\omega}, F \in \mathcal{F}\}$

$\langle a, F\rangle < \langle b, H\rangle$ if $b \sqsubseteq a$, $F \subset H$, and $a \setminus b \subset H$.

Definition (Mathias real for $\mathcal{F}$)

$\mathbf{x} = \bigcup\{a \colon \langle a, F\rangle\} \in \mathbf{G}$, where $\mathbf{G}$ is an $\mathbb{M}(\mathcal{F})$ generic filter.

Fact (1)

A Mathias real is a pseudo-intersection of $\mathcal{F}$ ($\mathbf{x} \subseteq^* F$ for each $F \in \mathcal{F}$).

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Fact (2)

For each $\uparrow$-cover O of $\mathcal{F}$ in V; $\mathbf{x} \in \bigcup O$.

Definition (pseudo-Mathias real for $\mathcal{F}$)

A set $m \subset \omega$ is a *pseudo-Mathias real for* $\mathcal{F}$ *over* V if

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Theorem

If $m \subset \omega$ is a pseudo-Mathias real for $\mathcal{F}$ and $c \subset \omega$ is a Cohen real, then $m \cap c$ is a Mathias real for $\mathcal{F}$.

Rothberger property

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$$\mathbf{S}_1(\mathcal{O}, \mathcal{O}) \quad \text{vs.} \quad \mathbf{S}_1(\uparrow\mathcal{O}, \mathcal{O})$$

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Theorem

Let $\mathcal{F}$ be a filter. $\mathbb{M}(\mathcal{F})$ does not add an eventually different real iff $\mathbf{S}_1(\uparrow\mathcal{O}, \mathcal{O})$.

ED real

$e \in \omega^\omega$ is an *eventually different (ED) real* over V iff
$(\forall f \in \omega^\omega \cap V)\,(\forall^\infty n \in \omega)\ \ e(n) \neq f(n)$

Rothberger game

Theorem (Pawlikovski)

The following are equivalent;

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Proposition (folklore / Bartoszyński / Miller)

There is an eventually different real in $V[G]$
iff $V[G] \models \omega^\omega \cap V$ *is meager.*