Open Covers and Selection principles Using Ideals and its Consequences

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Basic notions

- ***Ideal***: A hereditary family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ ($B \in \mathcal{I}$ for any $B \subseteq A \in \mathcal{I}$) that is closed under unions ($A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$), contains all finite subsets of ω and $\omega \notin \mathcal{I}$.

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- ***Filter:*** For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote $\mathcal{A}^d = \{\omega \setminus A : A \in \mathcal{A}\}$. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a **filter** if $\mathcal{F}^d$ is an ideal.

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- ***Filter:*** For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote $\mathcal{A}^d = \{\omega \setminus A : A \in \mathcal{A}\}$. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a **filter** if $\mathcal{F}^d$ is an ideal.
- ***Associated Filter:*** If $\mathcal{I}$ is a proper ideal in Y (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a *filter* in Y.
  • It is called the filter associated with the ideal $\mathcal{I}$.

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An open cover of X is said to be a γ-cover if each element of X belongs to all but finitely many elements of the cover. The set of all γ-covers will be denoted by Γ

• (Das, HJM, 2013)

Definition

Let $\mathcal{I}$ be an ideal. A countable open cover $\mathcal{U} = \{U_n : n \in \omega\}$ of a topological space X is said to be an **$\mathcal{I}$-γ-cover**, if for every n, $U_n \neq X$, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$. The family of all open $\mathcal{I}$-γ-covers of X will be denoted by $\mathcal{I}$-Γ.

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Definition

A countable open cover $\mathcal{U} = \{U_n : n \in \omega\}$ of X is said to be an $\mathcal{I}$-*large cover* if for each $x \in X$ the set $\{n \in \omega : x \in U_n\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-large covers will be denoted by $\mathcal{I}$-Λ.

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- (Das, Chandra, Samanta, submitted, 2017)

Definition

A countable open cover $\mathcal{U} = \{U_n : n \in \omega\}$ of X is said to be an *$\mathcal{I}$-τ-cover* if it is an $\mathcal{I}$-large cover and for each $x, y \in X$ $(x \neq y)$ either $\{n \in \omega : x \in U_n, y \notin U_n\} \in \mathcal{I}$ or $\{n \in \omega : x \notin U_n, y \in U_n\} \in \mathcal{I}$. $\mathcal{I}$-T will denote the set of all $\mathcal{I}$-τ-covers.

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- *$\mathcal{I}$-groupable cover: (Das, HJM, 2013)*

Definition

A countable open cover $\mathcal{U} = \{U_n : n \in \omega\}$ of X is said to be *$\mathcal{I}$-groupable* if it can be represented as a countable union of finite, pairwise disjoint subfamilies $\mathcal{V}_n, n \in \omega$, such that for each $x \in X$ the set $\{n \in \omega : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

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- *$\mathcal{I}$-Hurewicz property: (Das, HJM, 2013)*

Definition

A space X is said to have the *$\mathcal{I}$-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there are finite $\mathcal{V}_n \subset \mathcal{U}_n, n \in \omega$ such that each $x \in X$, $\{n \in \omega : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

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Theorem

If a Lindelöf space X satisfies $S_{fin}(\Omega, \mathcal{I} - \mathcal{O}^{gp})$ then X has the $\mathcal{I}$-Hurewicz property provided $S_1(\mathcal{F}(\mathcal{I}), \mathcal{F}(\mathcal{I}))$ holds (?).

• $(\mathcal{U}_n : n \in \omega)$ be a sequence of countable open covers of X. Let M be the set of all those positive integers n such that $\mathcal{U}_n$ contains a finite subset covering X. If $M \in \mathcal{F}(\mathcal{I})$ then there is nothing to prove.

• Otherwise write $(\mathcal{U}_n : n \in \omega \backslash M)$ as $(\mathcal{U}_n : n \in \omega)$. For each $n \in \omega$, take

$$\mathcal{V}_n = \{U_1 \cup U_2 \cup \cdots \cup U_k : U_i \in \mathcal{U}_n, i \leq k, k \in \omega\}.$$

• Each $\mathcal{V}_n$ is a countable ω-cover of X. Put

$$\mathcal{V}_n = \{V_{n,m} : m \in \omega\}, n \in \omega.$$

• Form $(\mathcal{W}_n : n \in \omega)$ of ω-covers of X as follows:

$$\mathcal{W}_1 = \mathcal{V}_1; \mathcal{W}_n = \{V_{1,m_1} \cap V_{2,m_2} \cap \ldots \cdots \cap V_{n,m_n} : n < m_1 < m_2 < \cdots < m_n$$

• Choose finite $\mathcal{P}_n \subset \mathcal{W}_n$ for each n so that $\mathcal{P} = \bigcup_{n \in \omega} \mathcal{P}_n$ is an $\mathcal{I}$-groupable cover of X.

- Write $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{H}_n$, where each $\mathcal{H}_n$ is finite subset of $\mathcal{P}$ and are pairwise disjoint from each other and for each $x \in X$, $\{n \in \omega : x \notin \bigcup \mathcal{H}_n\} \in \mathcal{I}$.

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• $A_k = \{n : \mathcal{H}_n \subset \bigcup_{i>k} \mathcal{P}_i\}$ for each $k \in \omega$. Since each $\mathcal{P}_n$ is finite and $\mathcal{H}_n$'s are pairwise disjoint from each other, every $A_k \in \mathcal{F}(\mathcal{I})$

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• By $S_1(\mathcal{F}(\mathcal{I}), \mathcal{F}(\mathcal{I}))$ choose $k_i \in A_i$ for $i = 1, 2, 3, \ldots$ such that $\{k_1 < k_2 < k_3 < \ldots\} \in \mathcal{F}(\mathcal{I})$.

• Let $\mathcal{G}_1$ denote the set of all $V_{1,p}$, the first components of elements of $\mathcal{H}_{k_1}$, $\mathcal{G}_2$ denote the set of all $V_{2,p}$, the second components of the elements of $\mathcal{H}_{k_2}$ and so on.

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• For each $G \in \mathcal{G}_n$ there is a finite set $\mathcal{U}_n^*(G) \subset \mathcal{U}_n$ with $G = \bigcup \mathcal{U}_n^*(G)$.

• Put $\mathcal{C}_n = \bigcup_{G\in\mathcal{G}_n} \mathcal{U}_n^*(G)$ which witnesses $\mathcal{I}$-Hurewicz property.

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Definition

An ideal $\mathcal{I}$ is said to be a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of elements of $\mathcal{I}$ there exists $A_\infty \in \mathcal{I}$ such that $A_n \backslash A_\infty \in [\omega]^{<\omega}$ $n \in \omega$. Also recall that after identifying the power set $\mathcal{P}(\omega)$ with the Cantor space $C = 2^\omega$ in a standard manner we may consider an ideal as a subset of C. An ideal is called an *analytic ideal* if it corresponds to an analytic subset of C.

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Definition

Let S be a set. We say that a map $\varphi : \mathcal{P}(S) \to [0, \infty]$ is a *submeasure* on S if it satisfies the following conditions:
- $\varphi(\emptyset) = 0$ and $\varphi(\{s\}) < \infty$ for every $s \in S$,
- φ is monotone: if $A \subset B \subset S$, then $\varphi(A) \leq \varphi(B)$,
- φ is subadditive: if $A, B \subset S$, then $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

A submeasure φ on $\mathbb{N}$ is *lower semicontimuous* if for every $A \subset \mathbb{N}$ we have $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [1, n])$.

• Note that a submeasure on $\mathbb{N}$ is lower semicontinuous if and only if it is lower semicontinuous as a function from $P(\omega)$ to $[0, \infty]$.

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• Solecki, APAL, 1999

Theorem

$\mathcal{I}$ is an analytic P-ideal if and only if it can be presented as

$$\mathcal{I} = Exh(\varphi) = \{A \subset \omega \ : \ \lim_{n\to\infty} \varphi(A \setminus [1, n]) = 0\} \qquad (*)$$

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• How many distinct analytic P-ideals are there ? Many ((Balcerzak - Das - Filipzak - Swaczina, AMH, 2015) or (Bartoszewich, Das, Glab, LAA, 2015)

Definition

For $f, g \in \omega^\omega$, $f \leq_{\mathcal{I}} g$ if and only if $\{n \in \omega : g(n) < f(n)\} \in \mathcal{I}$ and $X \subset \omega^\omega$ is $\mathcal{I}$-bounded if there is a function $g \in \omega^\omega$ such that $x \leq_{\mathcal{I}} g$ for each $x \in X$. The minimal cardinality of an unbounded subset of ω^ω w.r.to $\leq_{\mathcal{I}}$ is denoted by $\mathfrak{b}(\mathcal{I})$.

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• $\mathfrak{b}(\mathcal{I}) \geq \mathfrak{b}$ [Filipów - Staniszewski, JMAA, 2015]
• Let $\overline{\omega} = \omega \cup \{\infty\}$ be the one-point compactification of ω and let $\overline{\omega}^\omega$ be the Tychonoff product of $\overline{\omega}$. Let $\mathcal{I}$ be an admissible proper ideal of ω and $\mathcal{F}(\mathcal{I})$ be the dual filter of $\mathcal{I}$. An element $f \in \overline{\omega}^\omega$ is said to be *finite in filter* if $\{n \in \omega : f(n) < \infty\} \in \mathcal{F}(\mathcal{I})$. Let $\mathbb{FIF}(\mathcal{I})$ be the subspace of $\overline{\omega}^\omega$ consisting of all functions which are finite in filter. The partial order $\leq_{\mathcal{I}}$ can be extended to $\mathbb{FIF}(\mathcal{I})$ in the natural way.

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• $\mathfrak{b}(\mathcal{I}) \geq \mathfrak{b}$ [Filipów - Staniszewski, JMAA, 2015]
• Let $\varpi = \omega \cup \{\infty\}$ be the one-point compactification of ω and let ϖ^ω be the Tychonoff product of ϖ. Let $\mathcal{I}$ be an admissible proper ideal of ω and $\mathcal{F}(\mathcal{I})$ be the dual filter of $\mathcal{I}$. An element $f \in \varpi^\omega$ is said to be *finite in filter* if $\{n \in \omega : f(n) < \infty\} \in \mathcal{F}(\mathcal{I})$. Let $\mathbb{FI}\mathcal{F}(\mathcal{I})$ be the subspace of ϖ^ω consisting of all functions which are finite in filter. The partial order $\leq_{\mathcal{I}}$ can be extended to $\mathbb{FI}\mathcal{F}(\mathcal{I})$ in the natural way.

Theorem

Let $A \subset \mathbb{FI}\mathcal{F}(\mathcal{I})$ has $\mathcal{I}$-Hurewicz property. Then A is $\mathcal{I}$-bounded.

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Let X be a zero-dimensional, separable metrizable space. Then X has $\mathcal{I}$-Hurewicz property if and only if every continuous image of X into ω^ω is $\mathcal{I}$-bounded.

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For a Hausdorff, Lindelöf space X and a meager ideal $\mathcal{I}$, X satisfies $S_{fin}(\Lambda, \mathcal{O}^{\mathcal{I}\text{-}gp})$ if and only if X has Hurewicz property.

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Let X be a Hausdorff, Lindelöf space . Then X satisfies $S_{fin}(\mathcal{I}\text{-}\Gamma, \mathcal{O}^{\mathcal{I}\text{-}gp})$ if and only if X has $\mathcal{I}$-Hurewicz property.

• Let $A = \{m \in \omega : x \notin \cup\mathcal{F}_m\} \in \mathcal{I}_d$. Now A has density zero implies $f(A)$ has also density zero as $|f(A)| \leq |A|$, i.e. $f(A) \in \mathcal{I}_d$. We prove that $B = \{n \in \omega : x \notin \cup_n\} \in \mathcal{I}_d$ by showing that $B \subseteq f(A)$. Now by definition of $_n$, $B = \{n \in \omega : x \notin \cup\mathcal{F}_m$ for all $m \in f^{-1}(n)\}$. Let $j \in B$. Then $x \notin \cup\mathcal{F}_m$ for all $m \in f^{-1}(j)$. Say $f^{-1}(j) = \{m_1, \cdots, m_k\}$, since f is finite to one. Thus $x \notin \cup\mathcal{F}_{m_i}, i = 1, \cdots, k$. Hence $\{m_1, \cdots, m_k\} \subset A$ and consequently $\{j\} = \{f(m_1), \cdots, f(m_k)\} \subset f(A)$ i.e. $j \in f(A)$. Thus $B \subseteq f(A)$ and so $B \in_d$. A closer look at the proof reveals that the proof actually uses the following property of the ideal $\mathcal{I}$.

• Let $A = \{m \in \omega : x \notin \cup \mathcal{F}_m\} \in \mathcal{I}_d$. Now A has density zero implies $f(A)$ has also density zero as $|f(A)| \leq |A|$, i.e. $f(A) \in \mathcal{I}_d$. We prove that $B = \{n \in \omega : x \notin \cup_n\} \in \mathcal{I}_d$ by showing that $B \subseteq f(A)$. Now by definition of $_n$, $B = \{n \in \omega : x \notin \cup \mathcal{F}_m$ for all $m \in f^{-1}(n)\}$. Let $j \in B$. Then $x \notin \cup \mathcal{F}_m$ for all $m \in f^{-1}(j)$. Say $f^{-1}(j) = \{m_1, \cdots, m_k\}$, since f is finite to one. Thus $x \notin \cup \mathcal{F}_{m_i}, i = 1, \cdots, k$. Hence $\{m_1, \cdots, m_k\} \subset A$ and consequently $\{j\} = \{f(m_1), \cdots, f(m_k)\} \subset f(A)$ i.e. $j \in f(A)$. Thus $B \subseteq f(A)$ and so $B \in_d$. A closer look at the proof reveals that the proof actually uses the following property of the ideal $\mathcal{I}$.

Lemma

For every finite to one mapping $f : \omega \to \omega$,
$f(\mathcal{I}) = \{f(A) : A \in \mathcal{I}\} \subseteq \mathcal{I}$

• Let $A = \{m \in \omega : x \notin \cup\mathcal{F}_m\} \in \mathcal{I}_d$. Now A has density zero implies $f(A)$ has also density zero as $|f(A)| \leq |A|$, i.e. $f(A) \in \mathcal{I}_d$. We prove that $B = \{n \in \omega : x \notin \cup_n\} \in \mathcal{I}_d$ by showing that $B \subseteq f(A)$. Now by definition of $_n$, $B = \{n \in \omega : x \notin \cup\mathcal{F}_m$ for all $m \in f^{-1}(n)\}$. Let $j \in B$. Then $x \notin \cup\mathcal{F}_m$ for all $m \in f^{-1}(j)$. Say $f^{-1}(j) = \{m_1, \cdots, m_k\}$, since f is finite to one. Thus $x \notin \cup\mathcal{F}_{m_i}, i = 1, \cdots, k$. Hence $\{m_1, \cdots, m_k\} \subset A$ and consequently $\{j\} = \{f(m_1), \cdots, f(m_k)\} \subset f(A)$ i.e. $j \in f(A)$. Thus $B \subseteq f(A)$ and so $B \in_d$. A closer look at the proof reveals that the proof actually uses the following property of the ideal $\mathcal{I}$.

Lemma

For every finite to one mapping $f : \omega \to \omega$, $f(\mathcal{I}) = \{f(A) : A \in \mathcal{I}\} \subseteq \mathcal{I}$

• which is satisfied by a large class of density ideals (Balcerzak - Das - Filipzak - Swaczina, AMH, 2015)

Theorem

I would like to Wish Prof. Marion Scheepers a Very Happy Sixtieth Birthday and many more fruitful years

THANK YOU FOR YOUR ATTENTION

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