Alster spaces are productively Menger

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Frontiers of Selection Principles,
Warsaw, 2017

Productively Lindelöf spaces

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For a topological property P, we say that a space X is **productively P** if, for every space Y that satisfies P, the product $X \times Y$ also satisfies P.

Alster spaces

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Question (Babinkostova–Pansera–Scheepers 2013)

Is it the case that X is Alster if and only if $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ holds?

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Let X be a space satisfying $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$. Then

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The following statements are equivalent:

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(b) there is a Michael space – i.e. a Lindelöf space whose product with ω^ω is not Lindelöf.

A bigger picture

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$\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$

$\downarrow$

Alster

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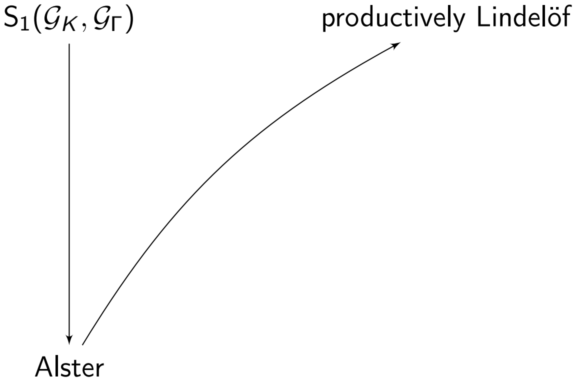

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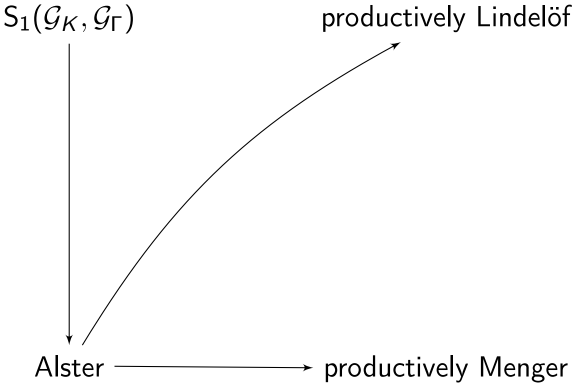

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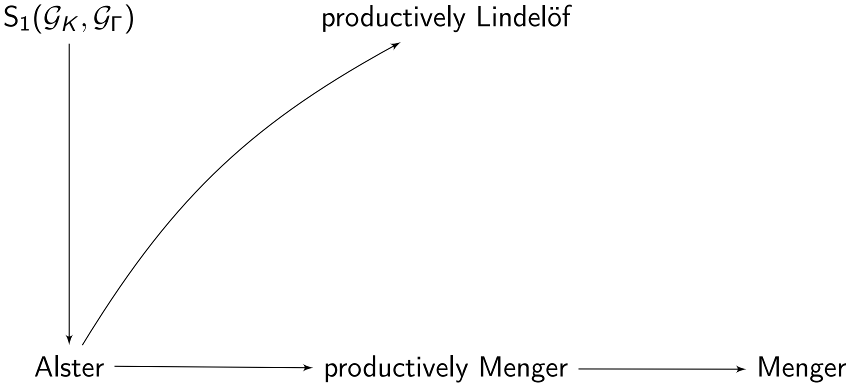

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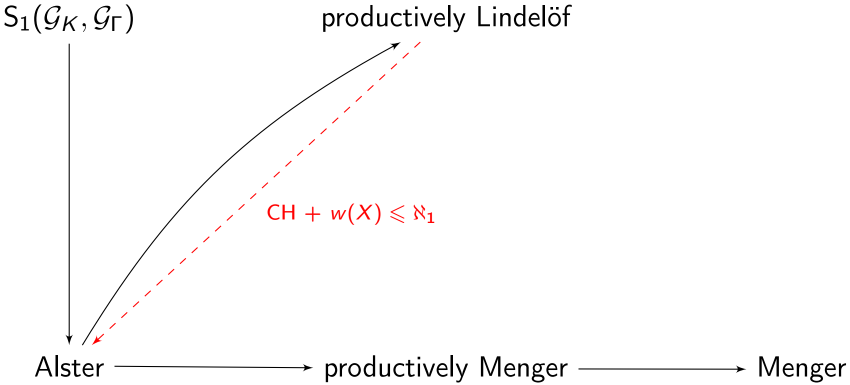

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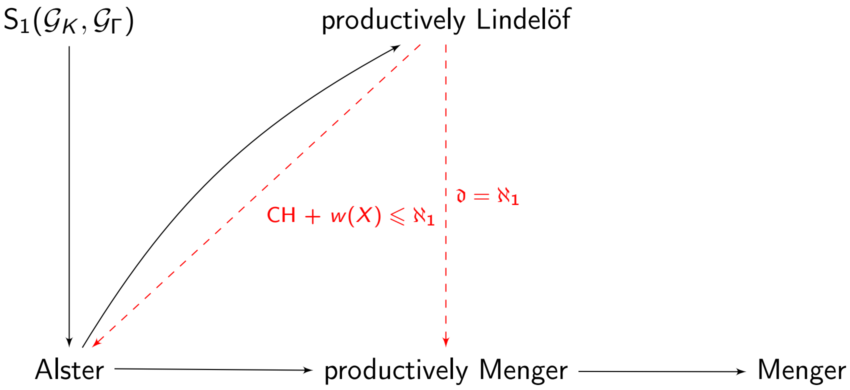

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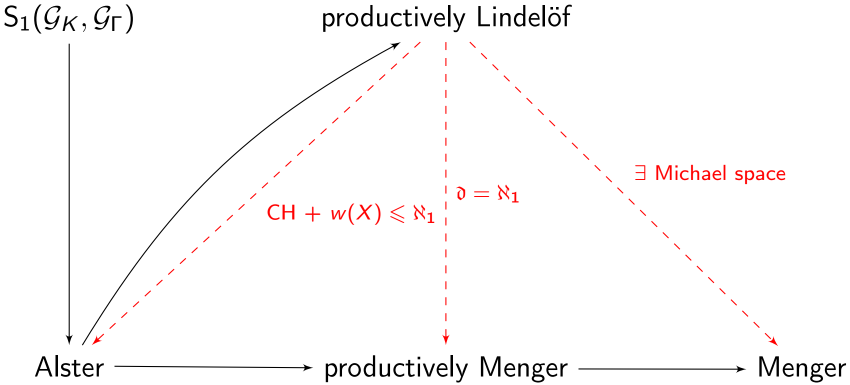

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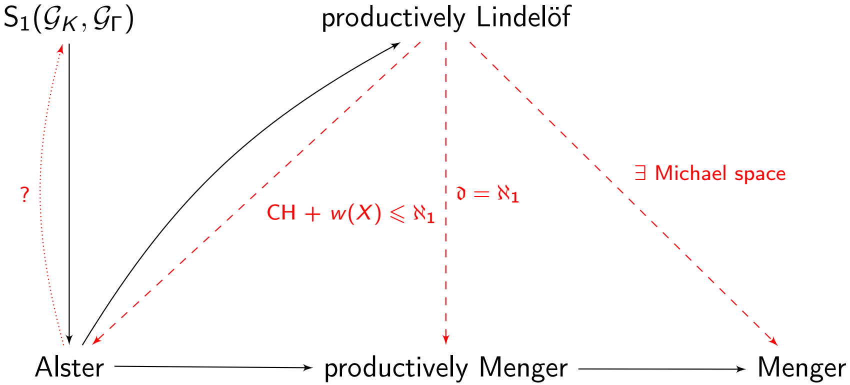

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If NO, then the first question has a negative answer as well.

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Dziękuję!

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