Winter School in Abstract Analysis
section Set Theory & Topology

27th Jan – 3th Feb 2018

TUTORIAL SPEAKERS

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REGISTRATION DEADLINE
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VENUE
Hejnice
Czech Republic

www.winterschool.eu

Background Photo: Ascension of St. Mary by A. Groll, interior of the Hejnice Church(c) 2017 Chris Le Sueur

Introduction to forcing for the working mathematician

lecture 1: Independence of the Continuum Hypothesis

David Chodounský

Institute of Mathematics CAS

Further reading

- K. Kunen, *Set theory; An introduction to independence proofs*, 1980
- S. Todorčević; I. Farah, *Some applications of the method of forcing*, 1995
- T. Jech, *Set Theory*, 2003
- T. Bartoszyński; H. Judah, *Set Theory: On the Structure of the Real Line*, 1995
- B. Balcar; T. Pazák; J. Verner, *An Exposition of Generic Extensions and Forcing in Set Theory*, preliminary version at `www.winterschool.eu/2009`

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- `www.google.com`

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Objects are sets. Language is $\langle =, \in \rangle$.

Other relations and operations are derived; $\subset, \cap, \cup, \mathcal{P}(\cdot), \emptyset, \ldots$

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Axioms:

1. Extensionality
   $\forall x \forall y \, (\forall z \, (z \in x \Leftrightarrow z \in y) \;\Leftrightarrow\; x = y)$
2. …

Infinitely many axioms, recursive system.

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Things provable in ZFC:

- If $2^{\aleph_n} < \aleph_\omega$ for all $n \in \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$ (Shelah 94),
- There exists an L-space (Moore 05),
- $\mathfrak{t} = \mathfrak{p}$ (Malliaris–Shelah 13),
- …

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Gödel's incompleteness implies there are undecidable sentences.

Continuum Hypothesis – CH

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Defined by Cantor in 1878.
Cantor believed it is true.
One of Hilbert's 23 problems (1900).
Gödel (1940) shows it cannot be disproved (if ZFC consistent).
Cohen (1963) shows it is independent of ZFC, invented *forcing*.

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- Idea 1: Remove some sets from V.
  Inner models, e.g. Gödel's constructible universe L.
- Idea 2: Add some new sets.
  The method of *forcing*.

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Theorem (Balcar–Vopěnka)

If $V[G] \models \mathrm{ZFC}$, then (essentially) P is a poset and G a V-generic filter.

Posets and Filters

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- We will assume that posets are nice; i.e.
  - antisymmetric
    $a \leq b \leq a \ \Rightarrow a = b$
  - separative
    $a \not\leq b \Rightarrow \exists c \leq a, c \perp b$
  - contains largest element **1**

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- $F \subset P$ is a filter if F is upwards closed and $\forall p, q \in F \, \exists r \in F$ such that $r \leq p, q$.
- $A \subset P$ is an antichain if $a, b \in A, a \neq b$ implies $a \perp b$.

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Focus on case 1.

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Choose P to be a poset of approximations of the desired G.

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Put $\quad f = \bigcup G$.

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For $\alpha \neq \beta \in \omega_2$ let $E_{(\alpha,\beta)} = \{\, f \in P : \exists n \in \omega \ \ f(\alpha, n) \neq f(\beta, n) \,\}$.
$E_{(\alpha,\beta)}$ is dense, i.e. $c_\alpha \neq c_\beta$.

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For $\alpha \in \omega_2$ define $c_\alpha \in 2^\omega$ by $c_\alpha(n) = \big(\bigcup G\big)(\alpha, n)$.
For $\alpha \neq \beta \in \omega_2$ let $E_{(\alpha,\beta)} = \{\, f \in P : \exists n \in \omega \;\; f(\alpha, n) \neq f(\beta, n) \,\}$.
$E_{(\alpha,\beta)}$ is dense, i.e. $c_\alpha \neq c_\beta$.
Put $C = \{\, c_\alpha : \alpha \in \omega_2 \,\}$, $V[G] \models |C| = |{\omega_2}^V|$.

Generic extensions (the forcing relation)

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- If $\mathcal{D}_\psi$ is dense, then $V[G] \models \psi$.
- $(\mathcal{D}_\psi \cup \mathcal{D}_{\neg\psi})$ is dense.
- If $\varphi \Rightarrow \psi$, then $\mathcal{D}_\varphi \subseteq \mathcal{D}_\psi$.

CH continued

$$P = \{ f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

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$G \cap D_c \neq \emptyset$ implies $G \cap \mathcal{D}_{\dot{c}=x} \neq \emptyset$ for some $x \in V$,
and $V[G] \models c = x$.

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$$P = \{ f : A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

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If P is c.c.c. and $V \models |\kappa| < |\lambda|$, then $V[G] \models |\kappa| < |\lambda|$.

Corollary

If P is c.c.c., then $V[G] \models |{\omega_2}^V| = \aleph_2$.

c.c.c. posets preserve cardinals

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Proof.

WLOG show that there is no surjection $b\colon \omega \to {\omega_1}^V$ in $V[G]$, other cases are analogous.

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Assume $b \colon \omega \to {\omega_1}^V$ in $V[G]$, investigate $\dot{b}$.

$\mathcal{D}_{\dot{b}(n)=\alpha} \perp \mathcal{D}_{\dot{b}(n)=\beta}$ for each $n \in \omega$ and $\alpha \neq \beta \in \omega_1$.

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Thus $R_n = \{\, \alpha : \mathcal{D}_{\dot{b}(n)=\alpha} \neq \emptyset \,\}$ is countable.

If $\mathcal{D}_{\dot{b}(n)=\alpha} = \emptyset$, then $V[G] \models b(n) \neq \alpha$.
I.e. $b(n) \in R_n$, and $\mathrm{Rng}(b) \subset \bigcup\{\, R_n : n \in \omega \,\}$.
$\bigcup\{\, R_n : n \in \omega \,\}$ is in V and countable, and b is not a surjection.

Δ-system lemma

Lemma

Suppose $\gamma \in \mathrm{On}$, $\{ a_\alpha : \alpha \in \omega_1 \} \subset [\gamma]^{<\omega}$*. There exists* $I \in [\omega_1]^{\omega_1}$ *and* $\Delta \in [\gamma]^{<\omega}$ *such that* $a_\alpha \cap a_\beta = \Delta$ *for each* $\alpha \neq \beta \in I$*.*

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Moreover if $\alpha < \beta$, $\chi \in a_\alpha \setminus \Delta$, $\xi \in a_\beta \setminus \Delta$, *then* $\chi < \xi$.