Scattered Spaces and Selections

Valentin Gutev

Department of Mathematics
Faculty of Science, University of Malta

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Zero-Selections

All spaces are Hausdorff topological spaces

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$\boxed{\tau_V}$ = "Vietoris topology on $\mathscr{F}(X)$"

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  - continuous = "τ_V-continuous"

Scattered Spaces

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X scattered $\iff$? continuous selection property for $\mathscr{F}(X)$

Continuous Zero-Selections

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- $X = (\omega_1 + 1) \vee_{\omega_1 = \omega_1} (\omega_1 + 1) = \overset{\omega_1}{\longmapsto} \overset{\omega_1}{\bullet} \overset{\omega_1}{\longleftarrow\!\!\mid}$ is a compact scattered space with a continuous selection for $\mathscr{F}(X)$, but has no continuous zero-selection.

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Compact $\longleftrightarrow$ τ_F = "Fell Topology on $\mathscr{F}(X)$"

X is an ordinal $\iff$ $\mathscr{F}(X)$ has a τ_F-continuous zero-selection

Two Open Questions:

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Let X be a space with a continuous zero-selection. Does there exist a coarser ordinal topology on X ?

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Let X be compact with a continuous $\varphi\colon (\mathscr{F}(X), \tau_V) \to (\mathscr{F}(X), \tau_V)$ such that $\varphi(S)$ are isolated points of S, $\forall S \in \mathscr{F}(X)$.

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Jan Pelant stated that the answer to Question 2 is “Yes”.

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Corollary 1 (X regular)

$$\exists \underbrace{f : \mathscr{F}(X) \to X}_{\text{continuous zero-selection}} \quad \underset{f = g \restriction \mathscr{F}(X)}{\Longleftarrow}\overset{g = f \circ \mathscr{cl}}{\Longrightarrow} \quad \exists \underbrace{g : \mathscr{A}(X) \to X}_{\text{continuous selection}}$$

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Corollary 2 (X regular)

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For a normal X, Corollary 2 was obtained by Artico, Marconi, Pelant, Rotter and Tkachenko, 2002.

An Inverse Problem

For a regular space X

$$\underbrace{f : \mathscr{F}(X) \to X}_{\text{continuous zero-selection}} \implies \mathscr{F}(Z) \text{ has a continuous selection, } \forall Z \in \mathscr{A}(X)$$

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Let X be a regular space such that $\mathscr{F}(Z)$ has a continuous selection, for every $Z \in \mathscr{A}(X)$. What can be said about X?

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Example (Many Selections $\not\Longrightarrow$ Zero-Selection)

- $X = (\omega_1 + 1) \vee_{\omega_1 = \omega} (\omega + 1)$ has no continuous zero-selection, but $\mathscr{F}(Z)$ has a continuous selection, $\forall Z \in \mathscr{A}(X)$.

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Example (scattered $\not\Longrightarrow$ true for each $Z \in \mathscr{A}(X)$)

- $X = (\omega_1 + 1) \vee_{\omega_1 = \omega} (\omega + 1)$ has no continuous zero-selection, but $\mathscr{F}(Z)$ has a continuous selection, $\forall Z \in \mathscr{A}(X)$.
- $X = (\omega_1 + 1) \vee_{\omega_1 = \omega_1} (\omega_1 + 1)$ has a continuous selection for $\mathscr{F}(X)$, but $\mathscr{F}(Z)$ has no continuous selection for some (uncountable) $Z \in \mathscr{A}(X)$ [Fujii-Miyazaki-Nogura, 2002].

Copies of $\mathbb{Q}$

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$\Uparrow$

A regular space X without isolated points (i.e. crowded)

$$\underbrace{\exists\, f : \mathscr{F}(X) \to X}_{\text{continuous selection}} \implies \mathbb{Q} \subset X$$

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Idea of Eric van Douwen, 1987

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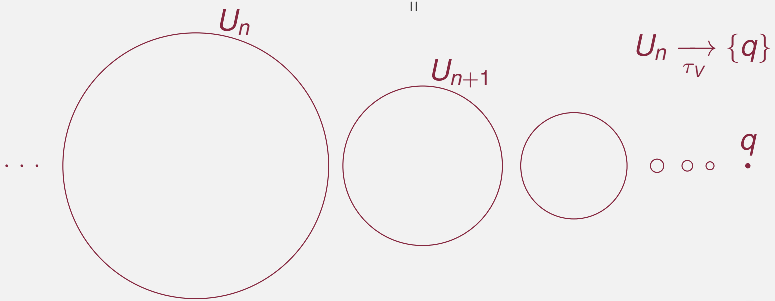

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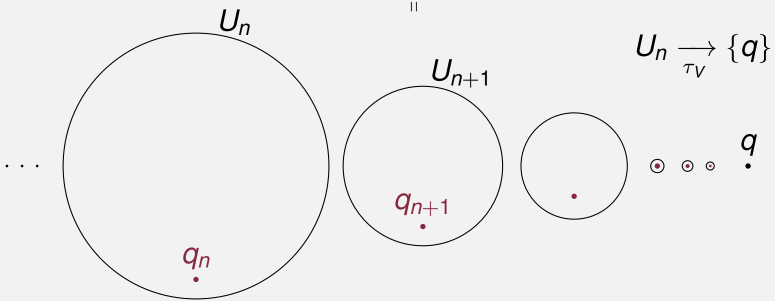

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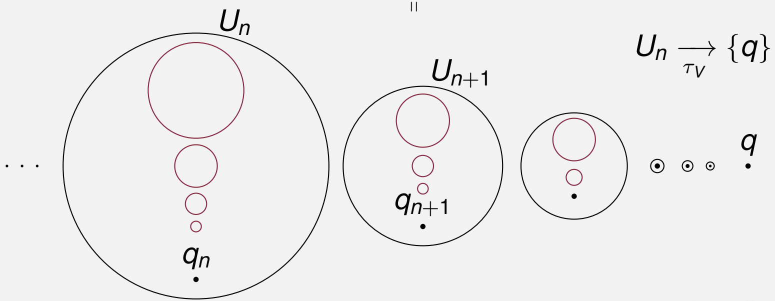

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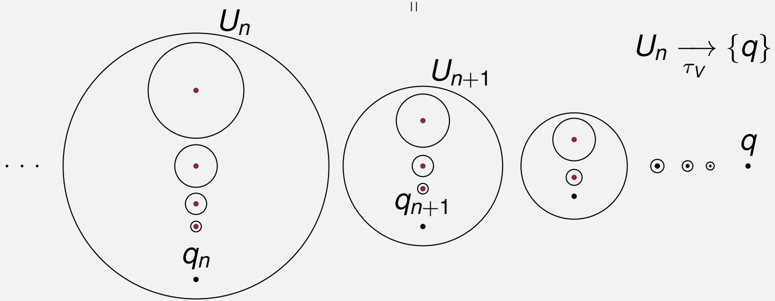

Disconnectedness-Like Properties

Compact Spaces

$$X \text{ compact} \quad \& \quad \exists\, \underbrace{g : \mathscr{F}(Z) \to Z}_{\text{continuous selection}},\ \forall\, Z \in \mathscr{A}(X)$$

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X scattered & orderable

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$$X \text{ scattered \& orderable} \implies \operatorname{ind}(X) = 0 \text{ (zero-dimensional)}$$

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R.C. Solomon, 1976

There exists a completely regular scattered space, which is not zero-dimensional.

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Question 4

Let X be a (completely) regular space such that $\mathscr{F}(Z)$ has a continuous selection, for every $Z \in \mathscr{A}(X)$. Then, is it true that X is zero-dimensional?

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Let X be a (completely) regular space such that $\mathscr{F}(Z)$ has a continuous selection, for every $Z \in \mathscr{A}(X)$. Then, is it true that X is zero-dimensional?

Observation 1

$\mathscr{F}(Z)$ has a continuous selection, $\forall Z \in \mathscr{A}(X)$ with $|X \setminus Z| \leq 1$. Then each connected component of X is compact.

Disconnectedness-Like Properties

Question 4

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Observation 2

$\mathscr{F}(Z)$ has a continuous selection, $\forall Z \in \mathscr{A}(X)$ with $|X \setminus Z| \leq 2$. Then X is totally disconnected.

Disconnectedness-Like Properties

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$\mathscr{F}(Z)$ has a continuous selection, $\forall Z \in \mathscr{A}(X)$ with $|X \setminus Z| \leq 2$.
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Observation 3

$\mathscr{F}(Z)$ has a continuous selection, for every open $Z \in \mathscr{A}(X)$.
Then X has a clopen π-base.

Disconnectedness-Like Properties

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$\mathscr{F}(Z)$ has a continuous selection, for every open $Z \in \mathscr{A}(X)$.
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Question 5

Let X be a (completely) metrizable space such that $\mathscr{F}(Z)$ has a continuous selection, for every G_δ-set $Z \in \mathscr{A}(X)$.

Then, is it true that $\dim(X) = 0$?