Menger-bounded and Scheepers-bounded sets in metrizable groups

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Background

Definition (M. Tkačenko 1998)

A topological group G is called o-bounded if for every sequences $U_0, U_1, \ldots$ of neighborhood of the identity in G, there exists a sequence $F_0, F_1, \ldots$ of finite subset of G such that $G = \bigcup_{n\in\omega} F_n \cdot U_n$.

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Let G and H be o-bounded groups. Is the product $G \times H$ o-bounded?

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Is the product of two o-bounded groups o-bounded?

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Contemporary definition

Definition

Assume that $(G, \cdot)$ is a topological group.

1. G is Menger-bounded if for each sequences $\{g \cdot U_n : g \in G\}$, $n \in \omega$ of open cover, there exist finite sets $F_n \subseteq G, n \in \omega$, such that $G = \bigcup_n F_n \cdot U_n$. Where $\{U_n\}_{n\in\omega}$ is a sequences of neighborhood of the unit.

Definition

A topological space X has the Menger property if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X there exists $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for all n such that $X = \bigcup\{\cup\mathcal{V}_n : n \in \omega\}$.

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Theorem (Babinkostova-Kočinac-Scheepers)

G is Scheeper-bounded if, and only if, G^k is Menger-bounded for all k.

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Theorem (Machura-Shelah-Tasban)

Under CH. There is a Menger-bounded group $G \leq \mathbb{Z}^\omega$ such that G^2 is not Menger-bounded.

Theorem (Mildenberger)

$\mathfrak{r} \geq \mathfrak{d}$ is a sufficient condition for the existence of subgroup of $\mathbb{Z}^\omega$ whose kth power is Menger-bounded but whose $(k+1)$st power is not.

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Is it consistent that for each Menger-bounded group G, G^2 is Menger-bounded?

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Hope

Theorem (L. Zdomskyy)

In the Miller model, the product of any two Menger spaces is Menger provided that the product is Lindelöf.

Subgroup of $\mathbb{Z}^\omega$

Theorem (Machura-Tasban)

Assume that G is subgroup of $\mathbb{Z}^\omega$.

1. *G is Menger-bounded if and only if there is $f \in \omega^\omega$ such that:*

$$\forall g \in G \exists^\infty n (|g||[0,n] \leq f(n)).$$

2. *G is Scheepers-bounded if and only if there is $f \in \omega^\omega$ such that:*

$$\forall \text{ finite } F \subseteq G \exists^\infty n \forall g \in F (|g||[0,n] \leq f(n)).$$

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Definition

Assume that $(G, \cdot)$ is a topological group and X is a subset of G.

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Assume that $X \subseteq \mathbb{Z}^\omega$.

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Assume that $X \subseteq \mathbb{Z}^\omega$.

1. *X is Menger-bounded, if, and only if, for each uniformly continuous image Y of X in ω^ω is not dominating.*
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Characterization of Scheepers-bounded under NCF

Definition

- Call a subset B of topological group G left totally bounded in G if for each neighborhood U of the identity, there is a finite $F \subseteq G$ such that $B \subseteq F \cdot U$.
- We call a topological group G is ω-narrow if for all open subset U, there are countable subset D, such that $D \cdot U = G$.

Theorem (NCF)

Let $(G, \cdot)$ is a ω-narrow metrizable topological group and $X \subseteq G$.

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The Answer

Theorem

There is a Menger-bounded set $X \subseteq \mathbb{Z}^\omega$ with X^2 is not Menger-bounded set. In particular, X can be choose as a G_δ subset.

Two operators on $[\omega]^\omega$

Definition

- Complement $c : A \longrightarrow A^c$
- Finite quotient $\cdot/h : A \longrightarrow A/h = h(A)$, where $h : \omega \longrightarrow \omega$ is an increasing function.

	preserved by Complement?	preserved by Finite quotient?
M set	Yes	Yes
MB set	No	No
SB set	No	$?$

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A filter $\mathcal{F}$ is called rapid filter if $\tilde{\mathcal{F}}$ is a dominating family in the mean of $\leq^*$.

Theorem

(1). (D. Fremlin) If $cov(\mathcal{M}) = \mathfrak{d}$, then there exists a rapid filter.

(2). (Mildenberger) NCF implies there is no rapid filter.

Theorem

The following are equivalent.

(1.) There is a rapid filter.

(2.) There is a Scheepers-bounded S and a finite-to-one function $h : \omega \to \omega$ such that S/h is dominating.

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The Answer

	preserved by Complement?	preserved by Finite quotient?
M set	Yes	Yes
MB set	No	No
SB set	No	$Yes \Longleftrightarrow No\, rapid\, filter$

A summary

	MB set	SB set	MB group	SB group
Union	$add(MB) = \mathfrak{b}$	$NCF \Rightarrow add(SB) = \mathfrak{d}$ $\mathfrak{d} \leq \mathfrak{r} \Rightarrow add(SB) = 2$	$add(\mathcal{MBG}) = \mathfrak{b}$	$NCF \Rightarrow add(SB) = \mathfrak{d}$ $\mathfrak{d} \leq \mathfrak{r} \Rightarrow add(SB) = 2$
Square	No	Yes	$\mathfrak{d} \leq \mathfrak{r} \Rightarrow No$?	Yes
Productive	No	$NCF \Rightarrow Yes$ $\mathfrak{d} \leq \mathfrak{r} \Rightarrow No$	$\mathfrak{d} \leq \mathfrak{r} \Rightarrow No$?	$NCF \Rightarrow Yes$ $\mathfrak{d} \leq \mathfrak{r} \Rightarrow No$

Thanks for your attention!

Happy 60th Birthday to professor Marion Scheepers