The 2-Rothberger Game and a Generalization

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The 2-Rothberger Game

Definition

The game $G_2(\mathcal{O}, \mathcal{O})$ is played as the familiar Rothberger game, except that II picks two elements instead of one.

Theorem

Let X be a T_2 space. Then the games $G_1(\mathcal{O}, \mathcal{O})$ and $G_2(\mathcal{O}, \mathcal{O})$ are equivalent on X.

Proof Outline

Two of the implications we must prove are trivial. If I $\uparrow G_2(\mathcal{O}, \mathcal{O})$, then I $\uparrow G_1(\mathcal{O}, \mathcal{O})$, and if II $\uparrow G_1(\mathcal{O}, \mathcal{O})$, then II $\uparrow G_2(\mathcal{O}, \mathcal{O})$.
Since we know that X is $S_1(\mathcal{O}, \mathcal{O})$ if and only if I $\not\uparrow G_1(\mathcal{O}, \mathcal{O})$ and that $S_1(\mathcal{O}, \mathcal{O})$ is equivalent to $S_k(\mathcal{O}, \mathcal{O})$ for $k \in \omega$, it is not difficult to prove the third:
I $\uparrow G_1(\mathcal{O}, \mathcal{O}) \Rightarrow X$ is not $S_1(\mathcal{O}, \mathcal{O}) \Rightarrow X$ is not $S_2(\mathcal{O}, \mathcal{O})$.
So if I $\uparrow G_1(\mathcal{O}, \mathcal{O})$, then I can simply play a sequence of open covers that witness the fact that X is not $S_2(\mathcal{O}, \mathcal{O})$ in order to win $G_2(\mathcal{O}, \mathcal{O})$.
So we need only show that II $\uparrow G_2(\mathcal{O}, \mathcal{O}) \Rightarrow$ II $\uparrow G_1(\mathcal{O}, \mathcal{O})$.

$(\text{II} \uparrow G_2(\mathcal{O},\mathcal{O}) \Rightarrow \text{II} \uparrow G_1(\mathcal{O},\mathcal{O}))$ – Setup

Let τ be a winning strategy for II in $G_2(\mathcal{O},\mathcal{O})$.
Let $g : \mathcal{O} \times X \to \mathcal{T}(X)$ be defined such that $x \in g(\mathcal{U}, x) \in \mathcal{U}$.
Where A is an infinite subset of ω, let $f(A) = \{min(A), min(A \setminus min(A))\}$.
Let $A_\varnothing = \omega$. For $s \in \omega^{<\omega}$, where A_s has been defined, let $(A_{s\frown i})_{i\in\omega}$ be a partition of $A_s \setminus f(A_s)$ such that when $2k \in A_{s\frown i}$, $2k+1 \in A_{s\frown i}$.
For $n \in \omega$, let $\phi(n) = s \Leftrightarrow n \in f(A_s)$.

$(\text{II} \uparrow G_2(\mathcal{O},\mathcal{O}) \Rightarrow \text{II} \uparrow G_1(\mathcal{O},\mathcal{O}))$ – Kickoff

Let $t_{<>}$ be the empty play of G_2.
Let $C_{t_{<>}} = \{x \in X : \forall \text{ open cover } \mathcal{U} \text{ of } X,\ x \in \overline{\bigcup \tau(t_{<>} \frown \mathcal{U})}\}$
$= \{x \in X : \forall \text{ open cover } \mathcal{U} \text{ of } X,\ x \in \overline{\bigcup \tau(\mathcal{U})}\}$.
We know that such a set has at most two elements.
If $|C_{t_{<>}}| = 2$, let $x_0^{<>}$ and $x_1^{<>}$ be its distinct elements.
If $|C_{t_{<>}}| = 1$, let $x_0^{<>} = x_1^{<>}$ be its lone element.
If $C_{t_{<>}} = \varnothing$, let $x_0^{<>}$ and $x_1^{<>}$ be arbitrary elements of X.
When I begins a play of $G_1(\mathcal{O},\mathcal{O})$ with an open cover $\mathcal{U}_0$ of X, let $\tau'(\mathcal{U}_0) = g(\mathcal{U}_0, x_0^{<>})$.
When I plays $\mathcal{U}_1$ next in $G_1(\mathcal{O},\mathcal{O})$, set $\tau'(\mathcal{U}_0, \tau'(\mathcal{U}_0), \mathcal{U}_1) = g(\mathcal{U}_1, x_1^{<>})$.

$s = <>$ Step

Suppose that we have a partial play
$t = \{\mathcal{U}_0, \tau'(\mathcal{U}_0), \mathcal{U}_1, \tau'(\mathcal{U}_0, \tau'(\mathcal{U}_0), \mathcal{U}_1)\}$.
Let $W_\varnothing(t) = g(\mathcal{U}_0, x_0^{<>}) \cup g(\mathcal{U}_1, x_1^{<>})$
(the union of the last two moves by II in t).
Let $Y_\varnothing(t) = X \setminus W_\varnothing(t)$.
Note that 1) X is Lindelöf and Y is a closed subset of X, so Y is also Lindelöf, and 2) $Y_\varnothing(t) \subseteq X \setminus C_{t_\varnothing}$.
The collection $\{X \setminus \overline{\bigcup \tau(\mathcal{U})} : \mathcal{U} \in \mathcal{O}\}$ is an open cover of $Y_\varnothing(t)$.
Since $Y_\varnothing(t)$ is Lindelöf, there is a countable subcollection $\mathcal{Q}_\varnothing(t)$ that covers $Y_\varnothing(t)$. So, for each $i \in \omega$, let $\mathcal{V}_{(i)}(t)$ be an open cover of X such that $\mathcal{Q}_\varnothing(t) = \{X \setminus \overline{\bigcup \tau(\mathcal{V}_{(i)}(t))} : i \in \omega\}$.

$s = <>$ Step, Continued

Set $t_{(i)}(t) = t_{\varnothing} \frown \mathcal{V}_{(i)}(t) \frown \tau(t_{\varnothing} \frown \mathcal{V}_{(i)}(t)) = (\mathcal{V}_{(i)}(t), \tau(\mathcal{V}_{(i)}(t))$, and set $C_{(i)}(t) = \{x \in X$:for each open cover $\mathcal{U}$ of X, $x \in \overline{\bigcup \tau(t_{(i)} \frown \mathcal{U})}\}$.
Define $x_0^{(i)}(t)$ and $x_1^{(i)}(t)$ as we did in the initial step.
Given a partial play t' extending t of length $m \in f(A_{(i)})$ innings, if I plays an open cover $\mathcal{U}_m$, set $\tau'(t' \frown \mathcal{U}_m) = g(\mathcal{U}_m, x_j^{(i)}(t))$, where $j = 1$ if $m = max(f(A_{(i)})$ and $j = 0$ otherwise.

Moving to the General Step

Suppose n is an even natural. For every even $m < n$, there is a strictly increasing finite sequence $(m_i)_{i\in\omega}$ of even naturals such that
(1) $m_0 = 0 < m_1 < ... < m_{k-1} < m_k = m$,
(2) $\phi(m_{i+1}) \geq \phi(m_i)$, and
(3) $|\phi(m_{i+1})| = |\phi(m_i)| + 1$.
Suppose t is a partial play of G_1 with an even number n of innings, such that t has the form
$t = \{\mathcal{U}_0, \tau'(\mathcal{U}_0), ..., \mathcal{U}_{n-1}, \tau'(t\restriction_{n-1}\frown\mathcal{U}_{n-1})\}$.
For each $i \in \{1, ..., k\}$, we will define $W_{\phi(m_i)}(t\restriction_{m_i+1})$, $Y_{\phi(m_i)}(t\restriction_{m_i+1})$, and so on in a similar way to what we did in the $s =<>$ step. Note that $Y_{\phi(m_i)}(t\restriction_{m_i+1})$ will exclude all $W_{\phi(m_j)}(t\restriction_{m_i+1})$ for $j \leq i$, not just the last one.

The General Step, Continued

Given a play t' of length $l = min(f(A_{\phi(m)\frown i}))$ that extends t, if I plays an open cover $\mathcal{U}_l$ next, set $\tau'(t' \frown \mathcal{U}_l) = g(\mathcal{U}_l, x_0^{\phi(m)\frown i})$, and when an open cover $\mathcal{U}_{l+1}$ is played in the following inning, set
$\tau'(t' \frown \mathcal{U}_l \frown \tau'(t' \frown \mathcal{U}_l) \frown \mathcal{U}_{l+1}) = g(\mathcal{U}_{l+1}, x_1^{\phi(m)\frown i})$.

τ' is a Winning Strategy

Suppose that some point $x \in X$ is left uncovered by II in a play t in which he follows τ'.

Consider a partial play t_s of G_2 that corresponds to some subplay of t where x has not been covered yet (we can simply take t_s to be $t_\varnothing$).

x is in the corresponding Y_s, which is covered by $\mathcal{Q}_s$, so there exists $i \in \omega$ such that $x \in X \setminus \overline{\bigcup \tau(t_s \frown \mathcal{V}_{s \frown i})}$.

Choose such an i, and note that $t_{s \frown i}$ is an extension of t_s such that x has still not been covered.

Continuing in this way, we build a play of G_2 in which II follows τ yet loses – a contradiction.

Monotone and Coordinatewise Monotone Properties

Here, $\mathcal{P}$ will be a property that a sequence of open sets can have, such as "is an open cover" or "covers a dense subset of X".

1) Suppose $\mathcal{P}$ is such that whenever $(U_n)_{n\in\omega}$ satisfies $\mathcal{P}$ and $(V_n)_{n\in\omega}$ is a sequence such that for each $n \in \omega$, there is $m \in \omega$ such that $U_n \subseteq V_m$, then $(V_n)_{n\in\omega}$ also satisfies $\mathcal{P}$. We will call such a property "monotone".

2) Suppose $\mathcal{P}$ is such that whenever $(U_n)_{n\in\omega}$ satisfies $\mathcal{P}$ and $(V_n)_{n\in\omega}$ is a sequence such that for each $n \in \omega$, $U_n \subseteq V_m$, then $(V_n)_{n\in\omega}$ also satisfies $\mathcal{P}$. We will call such a property "coordinatewise monotone".

A property that is monotone will also be coordinatewise monotone. The property "is a γ-cover" is coordinatewise monotone but not monotone.

Rothberger and Point-Open Games with $\mathcal{P}$

Theorem

(i) $G_1(\mathcal{O}, \mathcal{P})$ is dual to the $\mathcal{P} - POG$.
(ii) If $\mathcal{P}$ is coordinatewise monotone and X is a compact T_2 space, then $G_2(\mathcal{O}, \mathcal{P})$ is dual to the $\mathcal{P} - 2 - POG$.

(i) follows from a proof similar to Galvin's for the original games. Tkachuk noted this for a specific property – that of covering a dense subset of X.

$(\mathrm{II} \uparrow G_2(\mathcal{O}, \mathcal{P}) \Rightarrow \mathrm{I} \uparrow \mathcal{P} - 2 - POG)$

Lemma

Suppose that X is a topological space and f is a function on the collection $\mathcal{O}$ such that $f(\mathcal{U})$ is an ordered pair (G_1, G_2) of elements of $\mathcal{U}$. If X is a compact T_2 space, then there exist $x_0, y_0 \in X$ such that for any open neighborhoods $U \ni x_0, V \ni y_0$ and for any open cover $\mathcal{U}$ of X, there is an open cover $\mathcal{V}$ of X such that $\mathcal{V}$ refines $\mathcal{U}$, $[f(\mathcal{V})]_0 \subseteq U$, and $[f(\mathcal{V})]_1 \subseteq U$.

When II has a winning strategy τ in $G_2(\mathcal{O}, \mathcal{P})$, we can form a winning strategy for I in $\mathcal{P} - 2 - POG$ by
*playing two such points (x_n, y_n) in each inning of the $\mathcal{P} - 2 - POG$,
*considering the neighborhoods (U_n, V_n) in II's response, then
*playing an open cover $\mathcal{U}_n$ in an ongoing play of $G_2(\mathcal{O}, \mathcal{P})$ where II follows his winning strategy, such that $\mathcal{U}_n$ refines the previous move (for $n \neq 0$) and each part of the response by τ is contained in U_n or V_n as applicable.

Showing that $(\mathrm{II} \uparrow \mathcal{P} - 2 - POG \Rightarrow \mathrm{I} \uparrow G_2(\mathcal{O}, \mathcal{P}))$ is nontrivial, but not especially interesting.
The remaining two implications follow from typical game arguments.

The End

Questions?
Comments?
Complaints?