

Ideals of nowhere dense sets in some topologies on integers

Marta Kwela

University of Gdańsk



Joint work with Prof. Andrzej Nowik

FRONTIERS OF SELECTION PRINCIPLES

Warsaw, Poland

22nd August 2017

Preliminaries

$\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of positive integers

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$ – the set of non-negative integers

For all $a, b \in \mathbb{N}$ the symbol $\{an + b\}$ stands for the infinite arithmetic progression with the initial term b and the difference a :

$$\{an + b\} = \{an + b : n \in \mathbb{N}_0\} = \{b, b + a, b + 2a, \dots\}.$$

(a, b) - the greatest common divisor of a and b

\mathbb{P} - the set of all prime numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of positive integers

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$ – the set of non-negative integers

For all $a, b \in \mathbb{N}$ the symbol $\{an + b\}$ stands for the infinite arithmetic progression with the initial term b and the difference a :

$$\{an + b\} = \{an + b : n \in \mathbb{N}_0\} = \{b, b + a, b + 2a, \dots\}.$$

(a, b) - the greatest common divisor of a and b

\mathbb{P} - the set of all prime numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of positive integers

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$ – the set of non-negative integers

For all $a, b \in \mathbb{N}$ the symbol $\{an + b\}$ stands for the infinite arithmetic progression with the initial term b and the difference a :

$$\{an + b\} = \{an + b : n \in \mathbb{N}_0\} = \{b, b + a, b + 2a, \dots\}.$$

(a, b) - the greatest common divisor of a and b

\mathbb{P} - the set of all prime numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of positive integers

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$ – the set of non-negative integers

For all $a, b \in \mathbb{N}$ the symbol $\{an + b\}$ stands for the infinite arithmetic progression with the initial term b and the difference a :

$$\{an + b\} = \{an + b : n \in \mathbb{N}_0\} = \{b, b + a, b + 2a, \dots\}.$$

(a, b) - the greatest common divisor of a and b

\mathbb{P} - the set of all prime numbers

Three topologies

One can consider three topologies on \mathbb{N} :

- Furstenberg's topology \mathcal{T}_F with the base $\mathcal{B}_F = \{\{an + b\} : b \leq a\}$;
- Golomb's topology \mathcal{T}_G with the base $\mathcal{B}_G = \{\{an + b\} : (a, b) = 1, b < a\}$;
- Kirch's topology \mathcal{T}_K with the base $\mathcal{B}_K = \{\{an + b\} : (a, b) = 1, b < a, a - \text{square-free}\}$.

All of these topologies have recently been studied by P. Szczuka.

Three topologies

One can consider three topologies on \mathbb{N} :

- Furstenberg's topology \mathcal{T}_F with the base $\mathcal{B}_F = \{\{an + b\} : b \leq a\}$;
- Golomb's topology \mathcal{T}_G with the base $\mathcal{B}_G = \{\{an + b\} : (a, b) = 1, b < a\}$;
- Kirch's topology \mathcal{T}_K with the base $\mathcal{B}_K = \{\{an + b\} : (a, b) = 1, b < a, a - \text{square-free}\}$.

All of these topologies have recently been studied by P. Szczuka.

Three topologies

One can consider three topologies on \mathbb{N} :

- Furstenberg's topology \mathcal{T}_F with the base $\mathcal{B}_F = \{\{an + b\} : b \leq a\}$;
- Golomb's topology \mathcal{T}_G with the base $\mathcal{B}_G = \{\{an + b\} : (a, b) = 1, b < a\}$;
- Kirch's topology \mathcal{T}_K with the base $\mathcal{B}_K = \{\{an + b\} : (a, b) = 1, b < a, a - \text{square-free}\}$.

All of these topologies have recently been studied by P. Szczuka.

Three topologies

One can consider three topologies on \mathbb{N} :

- **Furstenberg's topology** \mathcal{T}_F with the base $\mathcal{B}_F = \{\{an + b\} : b \leq a\}$;
- **Golomb's topology** \mathcal{T}_G with the base $\mathcal{B}_G = \{\{an + b\} : (a, b) = 1, b < a\}$;
- **Kirch's topology** \mathcal{T}_K with the base $\mathcal{B}_K = \{\{an + b\} : (a, b) = 1, b < a, a - \text{square-free}\}$.

All of these topologies have recently been studied by P. Szczuka.

Three topologies

One can consider three topologies on \mathbb{N} :

- **Furstenberg's topology** \mathcal{T}_F with the base $\mathcal{B}_F = \{\{an + b\} : b \leq a\}$;
- **Golomb's topology** \mathcal{T}_G with the base $\mathcal{B}_G = \{\{an + b\} : (a, b) = 1, b < a\}$;
- **Kirch's topology** \mathcal{T}_K with the base $\mathcal{B}_K = \{\{an + b\} : (a, b) = 1, b < a, a - \text{square-free}\}$.

All of these topologies have recently been studied by P. Szczuka.

Three ideals

An **ideal** on \mathbb{N} is a family of subsets of \mathbb{N} closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets.

Obviously, in any topology, the nowhere dense sets form an ideal. Let us then define three ideals on \mathbb{N} :

- Furstenberg's ideal \mathcal{I}_F of all nowhere dense sets in \mathcal{T}_F ;
- Golomb's ideal \mathcal{I}_G of all nowhere dense sets in \mathcal{T}_G ;
- Kirch's ideal \mathcal{I}_K of all nowhere dense sets in \mathcal{T}_K .

Three ideals

An **ideal** on \mathbb{N} is a family of subsets of \mathbb{N} closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets.

Obviously, in any topology, the nowhere dense sets form an ideal. Let us then define three ideals on \mathbb{N} :

- Furstenberg's ideal \mathcal{I}_F of all nowhere dense sets in \mathcal{T}_F ;
- Golomb's ideal \mathcal{I}_G of all nowhere dense sets in \mathcal{T}_G ;
- Kirch's ideal \mathcal{I}_K of all nowhere dense sets in \mathcal{T}_K .

Three ideals

An **ideal** on \mathbb{N} is a family of subsets of \mathbb{N} closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets.

Obviously, in any topology, the nowhere dense sets form an ideal. Let us then define three ideals on \mathbb{N} :

- **Furstenberg's ideal** \mathcal{I}_F of all nowhere dense sets in \mathcal{T}_F ;
- **Golomb's ideal** \mathcal{I}_G of all nowhere dense sets in \mathcal{T}_G ;
- **Kirch's ideal** \mathcal{I}_K of all nowhere dense sets in \mathcal{T}_K .

Three ideals

An **ideal** on \mathbb{N} is a family of subsets of \mathbb{N} closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets.

Obviously, in any topology, the nowhere dense sets form an ideal. Let us then define three ideals on \mathbb{N} :

- **Furstenberg's ideal** \mathcal{I}_F of all nowhere dense sets in \mathcal{T}_F ;
- **Golomb's ideal** \mathcal{I}_G of all nowhere dense sets in \mathcal{T}_G ;
- **Kirch's ideal** \mathcal{I}_K of all nowhere dense sets in \mathcal{T}_K .

Three ideals

An **ideal** on \mathbb{N} is a family of subsets of \mathbb{N} closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets.

Obviously, in any topology, the nowhere dense sets form an ideal. Let us then define three ideals on \mathbb{N} :

- **Furstenberg's ideal** \mathcal{I}_F of all nowhere dense sets in \mathcal{T}_F ;
- **Golomb's ideal** \mathcal{I}_G of all nowhere dense sets in \mathcal{T}_G ;
- **Kirch's ideal** \mathcal{I}_K of all nowhere dense sets in \mathcal{T}_K .

Three ideals

Example (P. Szczuka)

$\mathbb{P} \in \mathcal{I}_F$, but \mathbb{P} is dense in \mathcal{T}_G and \mathcal{T}_K (therefore it does not belong to \mathcal{I}_G nor \mathcal{I}_K).

Example (M. Kwela, A. Nowik)

The set of even numbers $\{2n + 2\}$ is in \mathcal{I}_G and \mathcal{I}_K , but it belongs to the base of \mathcal{T}_F (therefore $\{2n + 2\} \notin \mathcal{I}_F$).

Theorem (M. Kwela, A. Nowik)

\mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are $F_{\sigma\delta}$ but not F_σ ideals.

Three ideals

Example (P. Szczuka)

$\mathbb{P} \in \mathcal{I}_F$, but \mathbb{P} is dense in \mathcal{T}_G and \mathcal{T}_K (therefore it does not belong to \mathcal{I}_G nor \mathcal{I}_K).

Example (M. Kwela, A. Nowik)

The set of even numbers $\{2n + 2\}$ is in \mathcal{I}_G and \mathcal{I}_K , but it belongs to the base of \mathcal{T}_F (therefore $\{2n + 2\} \notin \mathcal{I}_F$).

Theorem (M. Kwela, A. Nowik)

\mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are $F_{\sigma\delta}$ but not F_σ ideals.

Three ideals

Example (P. Szczuka)

$\mathbb{P} \in \mathcal{I}_F$, but \mathbb{P} is dense in \mathcal{T}_G and \mathcal{T}_K (therefore it does not belong to \mathcal{I}_G nor \mathcal{I}_K).

Example (M. Kwela, A. Nowik)

The set of even numbers $\{2n + 2\}$ is in \mathcal{I}_G and \mathcal{I}_K , but it belongs to the base of \mathcal{T}_F (therefore $\{2n + 2\} \notin \mathcal{I}_F$).

Theorem (M. Kwela, A. Nowik)

\mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are $F_{\sigma\delta}$ but not F_σ ideals.

Topological representations

We show that our new ideals (on countable set) may be connected to some σ -ideals in separable metrizable spaces. This connection has been introduced by M. Sabok and J. Zapletal.

Definition (M. Sabok, J. Zapletal)

Suppose that X is a separable metrizable space, $D \subseteq X$ – a dense countable set, and I – σ -ideal on X containing all singletons. Then

$$\mathcal{I}_I = \{A \subseteq D : cl(A) \in I\}$$

is an ideal on D .

Given an ideal \mathcal{I} on \mathbb{N} we say that \mathcal{I} has a *topological representation* if there are I, D, X as above, for which \mathcal{I} is isomorphic to \mathcal{I}_I (i.e. there exists a bijection $f : \mathbb{N} \rightarrow D$ such that $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{I}_I$).

In such a case we say that \mathcal{I} is *represented on X by I* .

Topological representations

We show that our new ideals (on countable set) may be connected to some σ -ideals in separable metrizable spaces. This connection has been introduced by M. Sabok and J. Zapletal.

Definition (M. Sabok, J. Zapletal)

Suppose that X is a separable metrizable space, $D \subseteq X$ – a dense countable set, and I – σ -ideal on X containing all singletons. Then

$$\mathcal{I}_I = \{A \subseteq D : cl(A) \in I\}$$

is an ideal on D .

*Given an ideal \mathcal{I} on \mathbb{N} we say that \mathcal{I} has a **topological representation** if there are I, D, X as above, for which \mathcal{I} is isomorphic to \mathcal{I}_I (i.e. there exists a bijection $f : \mathbb{N} \rightarrow D$ such that $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{I}_I$).*

*In such a case we say that \mathcal{I} is **represented on X by I** .*

Topological representations

We show that our new ideals (on countable set) may be connected to some σ -ideals in separable metrizable spaces. This connection has been introduced by M. Sabok and J. Zapletal.

Definition (M. Sabok, J. Zapletal)

Suppose that X is a separable metrizable space, $D \subseteq X$ – a dense countable set, and I – σ -ideal on X containing all singletons. Then

$$\mathcal{I}_I = \{A \subseteq D : cl(A) \in I\}$$

is an ideal on D .

*Given an ideal \mathcal{I} on \mathbb{N} we say that \mathcal{I} **has a topological representation** if there are I, D, X as above, for which \mathcal{I} is isomorphic to \mathcal{I}_I (i.e. there exists a bijection $f : \mathbb{N} \rightarrow D$ such that $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{I}_I$).*

*In such a case we say that \mathcal{I} is **represented on X by I** .*

Topological representations

We show that our new ideals (on countable set) may be connected to some σ -ideals in separable metrizable spaces. This connection has been introduced by M. Sabok and J. Zapletal.

Definition (M. Sabok, J. Zapletal)

Suppose that X is a separable metrizable space, $D \subseteq X$ – a dense countable set, and I – σ -ideal on X containing all singletons. Then

$$\mathcal{I}_I = \{A \subseteq D : cl(A) \in I\}$$

is an ideal on D .

*Given an ideal \mathcal{I} on \mathbb{N} we say that \mathcal{I} **has a topological representation** if there are I, D, X as above, for which \mathcal{I} is isomorphic to \mathcal{I}_I (i.e. there exists a bijection $f : \mathbb{N} \rightarrow D$ such that $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{I}_I$).*

*In such a case we say that \mathcal{I} **is represented on X by I** .*

Definition

Let \mathcal{I} be an ideal on \mathbb{N} .

- (i) \mathcal{I} is *tall* if any infinite set in \mathbb{N} contains an infinite subset that belongs to \mathcal{I} .
- (ii) \mathcal{I} is *ω -+-diagonalizable* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$.
- (iii) \mathcal{I} is *countably separated* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ and $B \notin \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$.

Definition

Let \mathcal{I} be an ideal on \mathbb{N} .

- (i) \mathcal{I} is *tall* if any infinite set in \mathbb{N} contains an infinite subset that belongs to \mathcal{I} .
- (ii) \mathcal{I} is *ω -+-diagonalizable* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$.
- (iii) \mathcal{I} is *countably separated* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ and $B \notin \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$.

Definition

Let \mathcal{I} be an ideal on \mathbb{N} .

- (i) \mathcal{I} is *tall* if any infinite set in \mathbb{N} contains an infinite subset that belongs to \mathcal{I} .
- (ii) \mathcal{I} is *ω -+-diagonalizable* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$.
- (iii) \mathcal{I} is *countably separated* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ and $B \notin \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$.

Definition

Let \mathcal{I} be an ideal on \mathbb{N} .

- (i) \mathcal{I} is *tall* if any infinite set in \mathbb{N} contains an infinite subset that belongs to \mathcal{I} .
- (ii) \mathcal{I} is *ω -+-diagonalizable* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$.
- (iii) \mathcal{I} is *countably separated* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of \mathbb{N} such that for any $A \in \mathcal{I}$ and $B \notin \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$.

Theorem (M. Kwela, A. Nowik)

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are tall and ω -+-diagonalizable.

Conjecture (M. Sabok, J. Zapletal)

An ideal on a countable set has a topological representation if and only if it is tall and ω -+-diagonalizable.

Theorem (M. Kwela, A. Nowik)

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are tall and ω -+-diagonalizable.

Conjecture (M. Sabok, J. Zapletal)

An ideal on a countable set has a topological representation if and only if it is tall and ω -+-diagonalizable.

Topological representations

Theorem (A. Kwela, M. Sabok)

An ideal on a countable set has a topological representation if and only if it is tall and countably separated.

Theorem (M. Kwela, A. Nowik)

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are tall and countably separated.

Corollary

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K have a topological representation.

Topological representations

Theorem (A. Kwela, M. Sabok)

An ideal on a countable set has a topological representation if and only if it is tall and countably separated.

Theorem (M. Kwela, A. Nowik)

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are tall and countably separated.

Corollary

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K have a topological representation.

Topological representations

Theorem (A. Kwela, M. Sabok)

An ideal on a countable set has a topological representation if and only if it is tall and countably separated.

Theorem (M. Kwela, A. Nowik)

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K are tall and countably separated.

Corollary

Ideals \mathcal{I}_F , \mathcal{I}_G and \mathcal{I}_K have a topological representation.

Topological representations

So far, only two examples of ideals with topological representations have been studied, namely:

$$\text{NWD} = \{A \subseteq \mathbb{Q} \cap [0, 1] : cl(A) \text{ is meager}\},$$

$$\text{NULL} = \{A \subseteq \mathbb{Q} \cap [0, 1] : cl(A) \text{ is of Lebesgue measure zero}\},$$

which are the ideals represented on $[0, 1]$ by σ -ideals of meager sets and sets of Lebesgue measure zero, respectively.

In 2003 I. Farah and S. Solecki proved that NWD and NULL are non-isomorphic.

Problem

Are \mathcal{I}_F , \mathcal{I}_G , \mathcal{I}_K , NWD and NULL pairwise non-isomorphic?

Topological representations

So far, only two examples of ideals with topological representations have been studied, namely:

$$\text{NWD} = \{A \subseteq \mathbb{Q} \cap [0, 1] : cl(A) \text{ is meager}\},$$

$$\text{NULL} = \{A \subseteq \mathbb{Q} \cap [0, 1] : cl(A) \text{ is of Lebesgue measure zero}\},$$

which are the ideals represented on $[0, 1]$ by σ -ideals of meager sets and sets of Lebesgue measure zero, respectively.

In 2003 I. Farah and S. Solecki proved that NWD and NULL are non-isomorphic.

Problem

Are \mathcal{I}_F , \mathcal{I}_G , \mathcal{I}_K , NWD and NULL pairwise non-isomorphic?

So far, only two examples of ideals with topological representations have been studied, namely:

$$\text{NWD} = \{A \subseteq \mathbb{Q} \cap [0, 1] : cl(A) \text{ is meager}\},$$

$$\text{NULL} = \{A \subseteq \mathbb{Q} \cap [0, 1] : cl(A) \text{ is of Lebesgue measure zero}\},$$







which are the ideals represented on $[0, 1]$ by σ -ideals of meager sets and sets of Lebesgue measure zero, respectively.

In 2003 I. Farah and S. Solecki proved that NWD and NULL are non-isomorphic.

Problem

Are \mathcal{I}_F , \mathcal{I}_G , \mathcal{I}_K , NWD and NULL pairwise non-isomorphic?

Thank you!

-  Farah I., Solecki S., Two $F_{\sigma\delta}$ ideals, Proc. Amer. Math. Soc. **131**(6) (2003) 1971–1975.
-  Kwela M., Nowik A., Ideals of nowhere dense sets in some topologies on integers, preprint.
-  Kwela A., Sabok M., Topological representations, J. Math. Anal. Appl. **422** (2015) 1434–1446.
-  Sabok M., Zapletal J., Forcing properties of ideals of closed sets, J. Symbolic Logic **76**(3) (2011) 1075–1095.
-  Szczuka P., Connections between connected topological spaces on the set of positive integers, Cent. Eur. J. Math. **11**(5) (2013) 876–881.
-  Szczuka P., The connectedness of arithmetic progressions in Furstenberg's, Golomb's and Kirch's topologies, Demonstratio Math. **43**(4) (2010) 899–909.