A CONCEPTUAL PROOF OF THE HUREWICZ THEOREM ON THE MENGER GAME

PIOTR SZEWCZAK AND BOAZ TSABAN

Abstract. We provide a conceptual proof of Hurewicz's Theorem that Menger's property is equivalent to the lack of a winning strategy for the first player in the corresponding game.

1. Background

Definition 1.1. *Menger's property* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ asserts that for each sequence of open covers, $\mathcal{U}_1, \mathcal{U}_2, \dots$, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{F}_2, \dots$, such that $\bigcup_n \mathcal{F}_n$ is a cover of the space. The symbol O in our notations indicates that we are provided with open covers, and need to obtain an open covers. The general framework of properties of this kind, and their relations to other notions, constitute the field of *selection principles* [2]. Within this field, Menger's property is one of the most studied and applied properties.

Menger's game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ is a game for two players, Alice and Bob, with an inning per each natural number n. In each inning, Alice picks an open cover of the space and Bob picks finitely many members from this cover. Bob wins if the sets he picked throughout the game cover the space. If this is not the case, Alice wins.

If Alice does not have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, then $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ holds. The converse implication is a deep theorem of Hurewicz. Hurewicz's theorem is often used within selection principles, and also has applications in diverse contexts such as D-spaces [1] and in additive Ramsey theory [3]. We present here a conceptual proof.

2. Tail covers

The proof uses the same initial simplifications as in Scheepers's proof of Hurewicz's Theorem. The new ingredient is an appropriate notion that goes through induction, and thus eliminates the necessity to track the entire history.

Theorem 2.1 (Hurewicz)**.** *Let X be an $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ space. Alice does not have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.*

Proof. Let X be a space satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. Fix an arbitrary strategy for Alice in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. Towards the main part of the proof, we make several easy modifications of Alice's strategy.

If any of Alice's covers is finite, then Bob can get hold of this cover and win. We thus assume that all covers have no finite subcovers. Since the space X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, it is Lindelöf. Thus, we can restrict Bob's moves to countable subcovers of Alice's covers. We will show that Bob can win despite this restriction. Thus, we may assume that Alice's covers are countable.

Given that, we may assume that Alice's open covers are *increasing*, that is, of the form $\{U_1, U_2, \dots\}$ with $U_1 \subseteq U_2 \subseteq \cdots$, and that Bob chooses a *single* set in each move. Indeed, given a countable cover $\{U_1, U_2, \dots\}$, we can restrict Bob's possible moves to the form $\{U_1, U_2, \dots, U_n\}$ (for $n \in \mathbb{N}$). Since the goal is just to cover the space, we may pretend that Bob is provided covers of the form

$$\{U_1, U_1 \cup U_2, U_1 \cup U_2 \cup U_3, \dots\},$$

and if Bob chooses an element $U_1 \cup \dots \cup U_n$, reply to Alice with the legal move $\{U_1, U_2, \dots, U_n\}$. If Bob manages to cover the space, this is not due to the unions.

Finally, we may assume that for each reply $\{U_1, U_2, \dots\}$ (with $U_1 \subseteq U_2 \subseteq \cdots$) of Alice's strategy to a move U, we have $U = U_1$. Indeed, we can transform the given cover into the cover $\{U, U \cup U_1, U \cup U_2, \dots\}$. If Bob chooses U, we provide Alice with the answer U_1, and if he chooses $U \cup U_n$, we provide Alice with the answer U_n. The addition of U in the new strategy does not help covering more points among those not already covered by the set U.

With these simplifications, Alice's strategy is identified with a tree of open sets, as follows: Alice's initial move is an open cover $\{U_{(1)}, U_{(2)}, \dots\}$. If Bob replies $U_{(n)}$, then Alice's move is $\{U_{(n,1)}, U_{(n,2)}, \dots\}$. In general, if Bob replies U_σ, for $\sigma \in \mathbb{N}^k$, then Alice's move is an increasing open cover

$$\mathcal{U}_\sigma := \{U_{(\sigma(1),\dots,\sigma(k),1)}, U_{(\sigma(1),\dots,\sigma(k),2)}, \dots\},$$

with $U_\sigma = U_{(\sigma(1),\dots,\sigma(k),1)}$.

The proof will reduce to the following concept.

Definition 2.2. A countable cover $\mathcal{U}$ of a space X is a *tail cover* if the set of intersections of cofinite subsets of $\mathcal{U}$ is an open cover of X.

Equivalently, a cover $\{U_1, U_2, \dots\}$ is a tail cover if the family

$$\left\{ \bigcap_{n=1}^{\infty} U_n, \bigcap_{n=2}^{\infty} U_n, \dots \right\}$$

of intersections of cofinal segments of the cover is an open cover.

Lemma 2.3. *Let n be a natural number. Define $\mathcal{V}_n := \bigcup_{\sigma \in \mathbb{N}^n} \mathcal{U}_\sigma$. Then the family $\mathcal{V}_n$ is a tail cover of X.*

Proof. The proof is by induction on n.

The open cover $\mathcal{V}_1 = \mathcal{U}_{()}$ is increasing, and thus the set of cofinite intersections is again $\mathcal{V}_1$, an open cover of X.

Let n be a natural number. For brevity, enumerate $\mathcal{V}_n = \{V_1, V_2, \dots\}$, and

$$\mathcal{V}_{n+1} = \bigcup_n \{V_1^n, V_2^n, \dots\},$$

where $V_1^n = V_n$, and $V_1^n \subseteq V_2^n \subseteq \cdots$ for each n.

Assume that the family $\mathcal{V}_n$ is a tail cover of X. Let $\mathcal{V}$ be a cofinite subset of $\mathcal{V}_{n+1}$. Let

$$I := \{\, n \in \mathbb{N} : \{V_1^n, V_2^n, \dots\} \subseteq \mathcal{V} \,\}.$$

The set I is a cofinite subset of $\mathbb{N}$. For the natural numbers $n \in I^{\mathsf{c}}$, let m_n be the minimal natural number with $V_{m_n}^n \in \mathcal{V}$. Then

$$\bigcap \mathcal{V} = \bigcap_{n\in I} (V_1^n \cap V_2^n \cap \cdots) \cap \bigcap_{n\in I^{\mathsf{c}}} \bigcap (\{V_1^n, V_2^n, \dots\} \cap \mathcal{V}) = \bigcap_{n\in I} V_n \cap \bigcap_{n\in I^{\mathsf{c}}} V_{m_n}^n.$$

The set $\bigcap_{n\in I} V_n$ is an intersection of a cofinite subset of $\mathcal{V}_n$, and is thus open. Since the set I^{c} is finite, the set $\bigcap \mathcal{V}$ is open.

Let $x \in X$. For almost all natural numbers n, we have $x \in V_n$. For the finitely many exceptional numbers n, x belongs to almost all sets V_m^n. Thus, x belongs to all but finitely many members of the family $\mathcal{V}_{n+1}$; equivalently, to an intersection of a cofinite subset of $\mathcal{V}_{n+1}$. □

For each n, let $\mathcal{V}_n'$ be the set of intersections of cofinite subsets of $\mathcal{V}_n$. Applying the property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ to the sequence $\mathcal{V}_1', \mathcal{V}_2', \dots$, Bob obtains cofinite sets $\mathcal{W}_n \subseteq \mathcal{V}_n$ such that $X = \bigcup_n \bigcap \mathcal{W}_n$. In the n-th inning, Alice provides Bob with a cover that is an infinite subset of the family $\mathcal{V}_n$. Since the family $\mathcal{W}_n$ is cofinite in $\mathcal{V}_n$, Bob can choose an element $V_n \in \mathcal{V}_n \cap \mathcal{W}_n$. Then $X = \bigcup_n V_n$, and Bob wins.

This completes the proof of Hurewicz's Theorem. □

A cover of a space is *large* if each point is covered by infinitely many members of the cover. Let Λ be the family of all large covers of an ambient space X. It is known, and not difficult to prove, that $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O}) = \mathsf{S}_{\mathrm{fin}}(\Lambda, \Lambda)$. Pawlikowski proved that $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ is equivalent to Alice not having a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\Lambda, \Lambda)$. Pawlikowski's proof is much more technical than the one presented here for Hurewicz's Theorem, but the present proof does not extend in a trivial manner to the case of large covers. It remains open to find a simple proof that $\mathsf{S}_{\mathrm{fin}}(\Lambda, \Lambda)$ implies that Alice does not have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\Lambda, \Lambda)$. Such an achievement would also imply a simple proof of Pawlikowski's Theorem that $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ is equivalent to Alice's not having a winning strategy in the corresponding game.

References

[1] L. Aurichi, *D-spaces, topological games, and selection principles*, Topology Proceedings **36** (2010), 107–122.
[2] Wikipedia, *Selection Principle*, en.wikipedia.org/wiki/Selection_principle
[3] B. Tsaban, *Algebra, selections, and additive Ramsey theory*, Fundamenta Mathematicae, to appear. DOI: 10.4064/fm342-1-2017

Piotr Szewczak, Institute of Mathematics, Faculty of Mathematics and Natural Science College of Sciences, Cardinal Stefan Wyszyński University in Warsaw, Wóycickiego 1/3, 01–938 Warsaw, Poland, and Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel
E-mail address: p.szewczak@wp.pl
URL: http://piotrszewczak.pl

Boaz Tsaban, Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel
E-mail address: tsaban@math.biu.ac.il
URL: http://math.biu.ac.il/~tsaban