On special kind of ideals of the real line generated by partitions into measure null and first category sets.

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Frontiers of Selection Principles, Warsaw, August 30, 2017

The starting point

That is the starting point for the presentation:
A partition of the real line (vs. the Cantor set 2^ω) into the set of measure null and first category: $\mathbb{R} = M \cup N$, where $M \in \mathcal{M}$ and $N \in \mathcal{N}$, where $\mathcal{M}$ is the collection of all first category sets and $\mathcal{N}$ is the collection of all sets of measure zero. We can find such a partition with the property that M is an F_σ set and N is a G_δ set.

Definition

- For such partition we know that $M + M$ and $N + N$ have nonempty interior (it follows from the Steinhaus Theorem).

Example

It is known that there exists a partition of the real line into G_σ set N of measure zero and F_σ set M of first category such that $int(M + N) = \emptyset$.

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An easy observation

If M, N is such partition as above then $(M+N)^c$ is a set of measure null and first category.

The ideal

So, it is natural to define (and ask about its properties) the σ-ideal $\mathcal{I}_{\mathcal{N},\mathcal{M}}$ as the σ-ideal generated by the family of sets $(M+N)^c$, where $\{N, M\}$ are all partitions of $\mathbb{R}$ into sets G_δ of measure null and F_σ of first category, respectively.

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Technical lemmas

A technical lemma

If the set $(X + X^c)^c$ is nonempty then it is a coset of some subgroup in $\langle \mathbb{R}, + \rangle$.

The family Fix(...)

Recall the following definition from

J. Cichoń, A.Jasiński, A.Kamburelis, P.Szczepaniak, *On translations of subsets of the real line* Proceedings of The Amercian Mathematical Society, Vol 130, No 6 (2001), 1833–1842.

Suppose that J is a transitive invariant ideal of subsets of $\mathbb{R}$. For $X \subseteq \mathbb{R}$ define $Fix(X, J) = \{x \in \mathbb{R} : (X + x) \triangle X \in J\}$.

Notice that $Fix(X, J)$ is an additive subgroup of $\mathbb{R}$. Denote $Fix(X) = Fix(X, \{\emptyset\})$.

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An abstract theorem

Suppose that $\mathcal{F} \subseteq P(\mathbb{R})$ is a transitive family of nonempty subsets of the real line. Moreover, assume that $\forall_{F\in\mathcal{F}} F \cup -F \in \mathcal{F}$. Then the ideal generated by the family $(X + X^c)^c$ where $X \in \mathcal{F} \setminus \{\emptyset\}$ is the same as the ideal generated by $\{Fix(F) + x \colon F \in \mathcal{F}\}, x \in \mathbb{R}$.

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Let us cite result from

M.Michalski, Sz.Żeberski, *Some properties of $\mathcal{I}$ − Luzinsets*, Topology and its Applications 55 (2015) Vol 198, 122-135.

There exists a comeager null $R \subseteq \mathbb{R}$ and a perfect null set P such that $R + P \subseteq R$.
A little strenghtening of this result is needed: $R - P \subseteq R$. Then modifying the authors argument we obtain:
There exists a (Borel) partition of $\mathbb{R}$ into $M \in \mathcal{M}$ and $N \in \mathcal{N}$ such that $P \subseteq (M + N)^c$.
Corollary: The ideal $\mathcal{I}_{\mathcal{N},\mathcal{M}}$ contains (some) perfect sets.

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A little stronger property...

Analogical argument may also be used to prove that the ideal $\mathcal{I}_{\mathcal{N},\mathcal{M}}$ has so called the (P) property (an ideal $\mathcal{I}$ has the (P) property iff for each perfect P there exists a perfect $Q \subseteq P$ such that $Q \in \mathcal{I}$, see:

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Descriptive properties

Is is easy to see that the set $(M+N)^c$ is Π^1_1.

There are examples of a G_δ set $G \subseteq \mathbb{R}$ and a closed (perfect) set E such that $E+G$ is not Borel, see for example...

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The main problem

Problem

Is there a partition $\{N, M\}$ into a G_δ set of measure zero and an F_σ set of first category, resp. such that $M + N$ is not Borel?

Conclusions from the main conjecture

Suppose that the answer to the main problem is no (i.e. for $M \in F_\sigma \cap \mathcal{M}$, $N \in G_\delta \cap \mathcal{N}$, $\{M, N\}$ a partition, we have $M + N$ is Borel.)

Then $(N + M)^c$ is a coset of some proper Σ^1_1 subgroup of $\mathbb{R}$. We know from

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...that every proper Σ^1_1 subgroup of $\mathbb{R}$ is contained in the ideal $\mathcal{E}$ (ideal generated by closed Lebesgue null sets).

Assume the main conjecture is false

$\mathcal{I}_{\mathcal{N},\mathcal{M}} \subseteq \mathcal{E}$.

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This theorem has many improvements, for example:

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A counterexample

P.Eliaš, Problem 5.10

Let A be a proper analytic subgroup of $\mathbb{R}$ and let $x \in A$. Does there exist an F_σ set separating the group A from its coset $A + x$?

Answer: There is a counterexample:
There exists a proper Σ^1_1 subgroup of $\mathbb{R}$ and x_0 such that there is no F_σ set separating the group A from $A + x_0$.

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The last slide

Thank You
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Your Attention

Happy Birthday, Marion