Classification of selectors for sequences of dense of function spaces

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Introduction

$C(X)$ — a set of all real-valued continuous functions defined on a Tychonoff space X.

$C_p(X)$ a set $C(X)$ provided with the pointwise convergence topology.

Main Question

One of the typical problems related with $C_p(X)$ is to look for duality theorems.
i.e. Given a topological property $\mathcal{P}$ find a topological property $\mathcal{Q}$ such that
X has $\mathcal{P} \Leftrightarrow C_p(X)$ has $\mathcal{Q}$.

Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set.

• Then $S_1(\mathcal{A}, \mathcal{B})$ denote the following hypothesis:
For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

• The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denote the following hypothesis:
For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \subset A_n$ is finite, and $\bigcup\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

• $U_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \subset A_n$ is finite, and $\{\bigcup B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

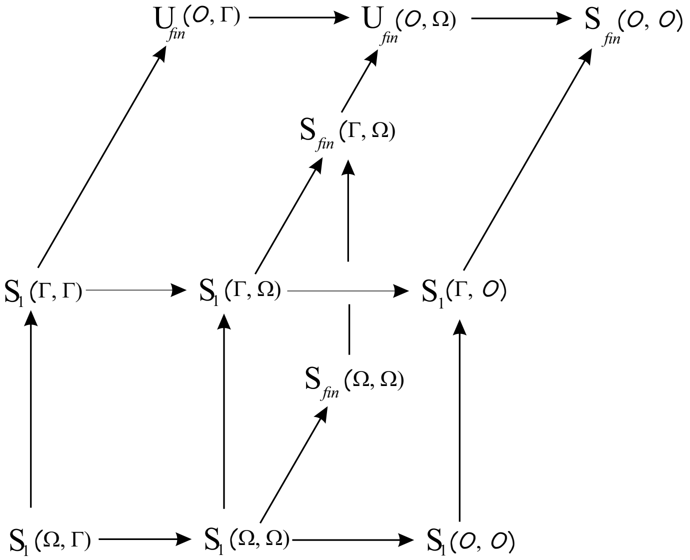

Fig. 1. The Scheepers Diagram

- $S_1(\mathcal{O}, \mathcal{O})$ denote the Rothberger property.

- $S_{fin}(\mathcal{O}, \mathcal{O})$ denotes the Menger property.

Question

What property of a space $C_p(X)$ characterizes the Rothberger property (the Menger property) of a space X ?

Main Definition

Definition

A set $A \subseteq C_p(X)$ will be called *n*-**dense** in $C_p(X)$, if for each n-finite set $\{x_1, \dots, x_n\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$ and an open sets $W_1, \dots, W_n$ in $\mathbb{R}$ there is $f \in A$ such that $f(x_i) \in W_i$ for $i \in \overline{1, n}$.

Main Definition

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A set $A \subseteq C_p(X)$ will be called n-**dense** in $C_p(X)$, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$ and an open sets $W_1, ..., W_n$ in $\mathbb{R}$ there is $f \in A$ such that $f(x_i) \in W_i$ for $i \in \overline{1, n}$.

Obviously, that if A is a n-dense set of $C_p(X)$ for each $n \in \omega$ then A is a dense set of $C_p(X)$.
For a space $C_p(X)$ we denote:
$\mathcal{A}_n$ — the family of a n-dense subsets of $C_p(X)$.
If $n = 1$, then we denote $\mathcal{A}$ instead of $\mathcal{A}_1$.

Main Definition

Definition

Let $f \in C(X)$. A set $B \subseteq C_p(X)$ will be called *n*-**dense at point** f, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ and $\epsilon > 0$ there is $h \in B$ such that $h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for $i \in \overline{1, n}$.

Main Definition

Definition

Let $f \in C(X)$. A set $B \subseteq C_p(X)$ will be called ***n*-dense at point** f, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ and $\epsilon > 0$ there is $h \in B$ such that $h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for $i \in \overline{1, n}$.

Obviously, that if B is a n-dense at point f for each $n \in \omega$ then $f \in \overline{B}$.

For a space $C_p(X)$ we denote:

$\mathcal{B}_{n,f}$ — the family of a n-dense at point f subsets of $C_p(X)$.

If $n = 1$, then we denote $\mathcal{B}_f$ instead of $\mathcal{B}_{1,f}$.

Let $\mathcal{U}$ be an open cover of X and $n \in \mathbb{N}$.

• $\mathcal{U}$ is an n-cover of X if for each $F \subset X$ with $|F| \leq n$, there is $U \in \mathcal{U}$ such that $F \subset U$.
• $\mathcal{U}$ is an ω-cover of X if for each finite $F \subset X$, there is $U \in \mathcal{U}$ such that $F \subset U$ (that is, $\mathcal{U}$ is an n-cover of X for each n).

• Ω — the family of open ω-covers of X.
• $\mathcal{O}_n$ — the family of open n-covers of X.

- $\mathcal{D}$ — the family of a dense subsets of $C_p(X)$.
- $\mathcal{A}_n$ — the family of a n-dense subsets of $C_p(X)$.
- $\mathcal{B}_{n,f}$ — the family of a n-dense at point f subsets of $C_p(X)$.
- $\mathcal{A} = \mathcal{A}_1$ and $\mathcal{B}_f = \mathcal{B}_{1,f}$.

Theorem 1.

For a space X, the following statements are equivalent:

1. $X \models S_1(\mathcal{O}, \mathcal{O})$ [Rothberger property];
2. $C_p(X) \models S_1(\mathcal{A}, \mathcal{A})$;
3. $C_p(X) \models S_1(\mathcal{D}, \mathcal{A})$;
4. $C_p(X) \models S_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$;
5. $C_p(X) \models S_1(\mathcal{B}_f, \mathcal{B}_f)$;
6. $C_p(X) \models S_1(\{\mathcal{B}_{n,f}\}_{n\in\mathbb{N}}, \mathcal{B}_f)$;
7. $C_p(X) \models S_1(\mathcal{A}, \mathcal{B}_f)$;
8. $C_p(X) \models S_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{B}_f)$.

M. Sakai

For each space X the following are equivalent.

1. $X^n \models S_1(\mathcal{O}, \mathcal{O})$ (X^n has Rothberger's property C'') for each $n \in \omega$.
2. $X \models S_1(\Omega, \Omega)$.

M. Scheeper

For each separable metric space X, the following are equivalent:

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$ [R-separable];
2. $X \models S_1(\Omega, \Omega)$.

R-separable

Corollary 1

For a space X, the following are equivalent:

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$ [R-separable];
2. $X \models S_1(\Omega, \Omega)$, and $iw(X) = \aleph_0$;
3. $X^n \models S_1(\mathcal{O}, \mathcal{O})$ for each $n \in \omega$, and $iw(X) = \aleph_0$.
4. $C_p(X) \models S_1(\{\mathcal{A}_n\}_{n\in\omega}, \mathcal{D})$;
5. $C_p(X^n) \models S_1(\mathcal{A}, \mathcal{A})$ for each $n \in \omega$.

Menger property

Theorem 2

For a space X, the following statements are equivalent:

1. $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ [Menger property];
2. $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{A})$;
3. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{A})$;
4. $C_p(X) \models S_{fin}(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$;
5. $C_p(X) \models S_{fin}(\mathcal{B}_f, \mathcal{B}_f)$;
6. $C_p(X) \models S_{fin}(\{\mathcal{B}_{n,f}\}_{n\in\mathbb{N}}, \mathcal{B}_f)$;
7. $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{B}_f)$;
8. $C_p(X) \models S_{fin}(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{B}_f)$.

M-separable

W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki

$X \models S_{fin}(\Omega, \Omega)$ iff $(\forall n \in \omega)\ X^n \models S_{fin}(\mathcal{O}, \mathcal{O})$.

A. Bella, M. Bonanzinga, M. Matveev

Corollary 2

For a space X, the following are equivalent:

1. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{D})$ [M-separable];
2. $X \models S_{fin}(\Omega, \Omega)$ and $iw(X) = \aleph_0$;
3. $(\forall n \in \omega)\ X^n \in S_{fin}(\mathcal{O}, \mathcal{O})$ and $iw(X) = \aleph_0$;
4. $C_p(X) \models S_{fin}(\{\mathcal{A}_n\}_{n \in \omega}, \mathcal{D})$;
5. $C_p(X^n) \models S_{fin}(\mathcal{A}, \mathcal{A})$ for each $n \in \omega$.

Part 2

Let $\mathcal{P}$ be a topological property.

A.V. Arhangel'skii calls X projectively $\mathcal{P}$ if every second countable continuous image of X is $\mathcal{P}$.

- Projectively Rothberger is projectively $S_1(\mathcal{O}, \mathcal{O})$.
- Projectively Menger is projectively $S_{fin}(\mathcal{O}, \mathcal{O})$.

Question

What property of a space $C_p(X)$ characterizes the projectively Rothberger property (projectively Menger property) of a space X?

Projectively Rothberger

M. Bonanzinga, F. Cammaroto, M. Matveev

The following conditions are equivalent for a space X:

1. X is projectively $S_1(\mathcal{O}, \mathcal{O})$ [projectively Rothberger];
2. every Lindelöf continuous image of X is Rothberger;
3. for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is Rothberger;
4. for every continuous mapping $f : X \mapsto \mathbb{R}$, $f(X)$ is Rothberger;
5. $X \models S_1(\mathcal{O}_{cz}^\omega, \mathcal{O}_{cz}^\omega)$.

- $\mathcal{O}_{cz}^\omega$ — the family of countable co-zero covers of X.

- $\mathcal{D}^\omega$ — the family of a countable dense subsets of $C_p(X)$.
- $\mathcal{A}^\omega_n$ — the family of a countable n-dense subsets of $C_p(X)$.
- $\mathcal{B}^\omega_{n,f}$ — the family of a countable n-dense at point f subsets of $C_p(X)$.
- $\mathcal{B}^\omega_f = \mathcal{B}^\omega_{1,f}$.

Theorem 3

For a space X, the following statements are equivalent:

1. X is projectively $S_1(\mathcal{O}, \mathcal{O})$ [projectively Rothberger];
2. $C_p(X) \models S_1(\mathcal{A}^\omega, \mathcal{A}^\omega)$;
3. $C_p(X) \models S_1(\mathcal{D}^\omega, \mathcal{A}^\omega)$;
4. $C_p(X) \models S_1(\{\mathcal{A}^\omega_n\}_{n\in\mathbb{N}}, \mathcal{A}^\omega)$;
5. $C_p(X) \models S_1(\mathcal{B}^\omega_f, \mathcal{B}^\omega_f)$;
6. $C_p(X) \models S_1(\{\mathcal{B}^\omega_{n,f}\}_{n\in\mathbb{N}}, \mathcal{B}^\omega_f)$;
7. $C_p(X) \models S_1(\mathcal{A}^\omega, \mathcal{B}^\omega_f)$.
8. $C_p(X) \models S_1(\{\mathcal{A}^\omega_n\}_{n\in\mathbb{N}}, \mathcal{B}^\omega_f)$.

Projectively Menger

M. Bonanzinga, F. Cammaroto, M. Matveev

The following conditions are equivalent for a space X:

1. X is projectively $S_{fin}(\mathcal{O}, \mathcal{O})$ [projectively Menger];
2. every Lindelöf continuous image of X is Menger;
3. for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is Menger;
4. for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is not dominating;
5. $X \models S_{fin}(\mathcal{O}^\omega_{cz}, \mathcal{O}^\omega_{cz})$.

Projectively $S_{fin}(\mathcal{O}, \mathcal{O})$

Theorem 4

For a space X, the following statements are equivalent:

1. X is projectively $S_{fin}(\mathcal{O}, \mathcal{O})$ [projectively Menger];
2. $C_p(X) \models S_{fin}(\mathcal{A}^\omega, \mathcal{A}^\omega)$;
3. $C_p(X) \models S_{fin}(\mathcal{D}^\omega, \mathcal{A}^\omega)$;
4. $C_p(X) \models S_{fin}(\{\mathcal{A}^\omega_n\}_{n\in\mathbb{N}}, \mathcal{A}^\omega)$;
5. $C_p(X) \models S_{fin}(\mathcal{B}^\omega_f, \mathcal{B}^\omega_f)$;
6. $C_p(X) \models S_{fin}(\{\mathcal{B}^\omega_{n,f}\}_{n\in\mathbb{N}}, \mathcal{B}^\omega_f)$;
7. $C_p(X) \models S_{fin}(\mathcal{A}^\omega, \mathcal{B}^\omega_f)$;
8. $C_p(X) \models S_{fin}(\{\mathcal{A}^\omega_n\}_{n\in\mathbb{N}}, \mathcal{B}^\omega_f)$.

Part 3

Lj. D. R. Kočinac

• A space is Rothberger iff it is Lindelöf and projectively Rothberger.
• A space is Menger iff it is Lindelöf and projectively Menger.

• $C_p(X) \models S_1(\mathcal{A}, \mathcal{A})$ iff $C_p(X) \models S_1(\mathcal{A}^\omega, \mathcal{A}^\omega) + ?$
• $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{A})$ iff $C_p(X) \models S_{fin}(\mathcal{A}^\omega, \mathcal{A}^\omega) + ?$

Lj. D. R. Kočinac

• A space is Rothberger iff it is Lindelöf and projectively Rothberger.
• A space is Menger iff it is Lindelöf and projectively Menger.

• $C_p(X) \models S_1(\mathcal{A}, \mathcal{A})$ iff $C_p(X) \models S_1(\mathcal{A}^\omega, \mathcal{A}^\omega)$ + ?
• $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{A})$ iff $C_p(X) \models S_{fin}(\mathcal{A}^\omega, \mathcal{A}^\omega)$ + ?

Definition

• $C_p(X)$ has the property (X_L), if every 1-dense set of $C_p(X)$ contains a countable 1-dense subset of $C_p(X)$.

Proposition 1

$C_p(X) \models (X_L)$ iff X is Lindelöf.

Proposition 2

- $C_p(X) \models S_1(\mathcal{A}, \mathcal{A})$ iff $C_p(X) \models (X_L) + C_p(X) \models S_1(\mathcal{A}^\omega, \mathcal{A}^\omega)$.

[A space is Rothberger iff it is Lindelöf and projectively Rothberger.]

- $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{A})$ iff $C_p(X) \models (X_L) + C_p(X) \models S_{fin}(\mathcal{A}^\omega, \mathcal{A}^\omega)$.

[A space is Menger iff it is Lindelöf and projectively Menger.]

Part 4

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$.

- $\mathcal{D}$ — the family of a dense subsets of $C_p(X)$;
- $\mathcal{S}$ — the family of a sequentially dense subsets of $C_p(X)$;
- $\mathcal{A}$ — the family of a 1-dense subsets of $C_p(X)$.

Recall that $U_{fin}(\mathcal{S}, \mathcal{D})$ is the selection hypothesis: whenever $\mathcal{U}_1$, $\mathcal{U}_2, ... \in \mathcal{S}$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{D}$.

For a function space $C(X)$, we can represent the condition $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{D}$ as $\forall\ f \in C(X)$ $\forall$ a base neighborhood $O(f) =< f, K, \epsilon >$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \omega$ such that for each $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Recall that $U_{fin}(\mathcal{S}, \mathcal{S})$ is the selection hypothesis: whenever $\mathcal{U}_1$, $\mathcal{U}_2, ... \in \mathcal{S}$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{S}$.

For a function space $C(X)$, we can represent the condition $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{S}$ as $\forall$ $f \in C(X)$ $\forall$ a base neighborhood of f $O(f) =< f, K, \epsilon >$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \omega$ such that for each $n > n'$ and $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_n$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

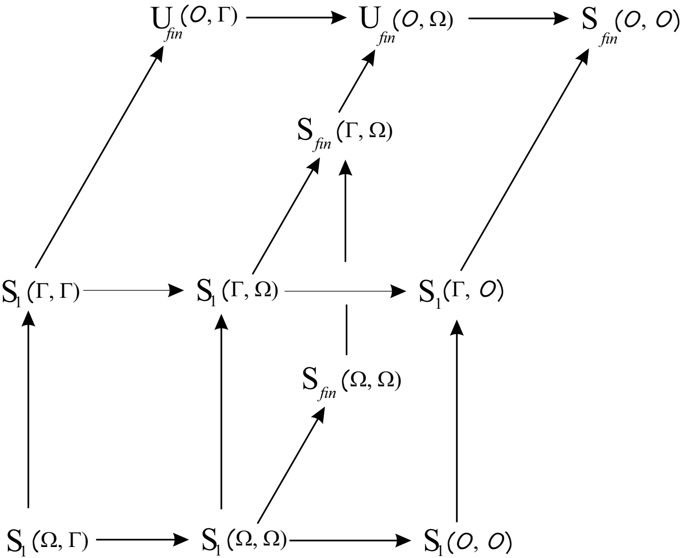
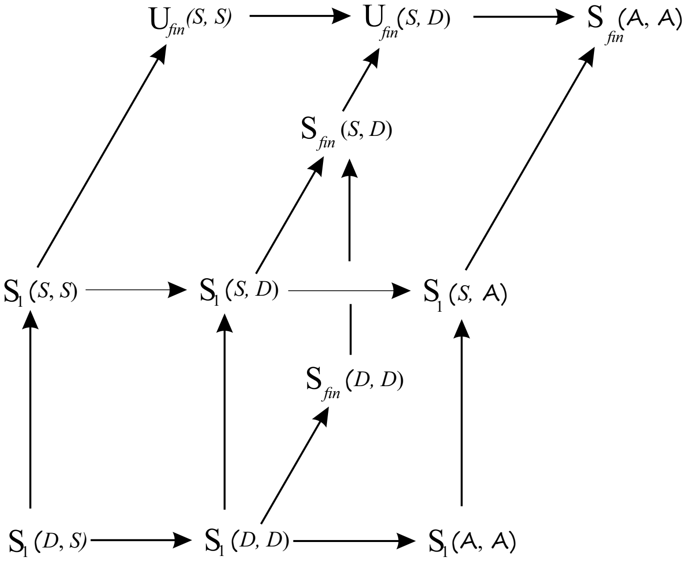


Fig. 1. The Scheepers Diagram



U_fin(S, S)
U_fin(S, D)
S_fin(A, A)
S_fin(S, D)
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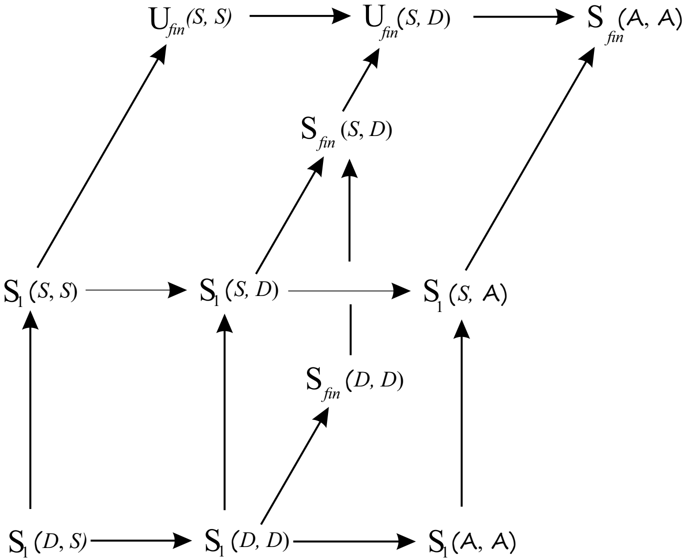


Fig. 2. The Diagram of selectors for sequences of dense sets of $C_p(X)$

Definition.(Sakai)

An γ-cover $\mathcal{U}$ of co-zero sets of X is γ_F-**shrinkable** if there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote:

- Γ_F — the family of γ_F-shrinkable γ-covers of X.

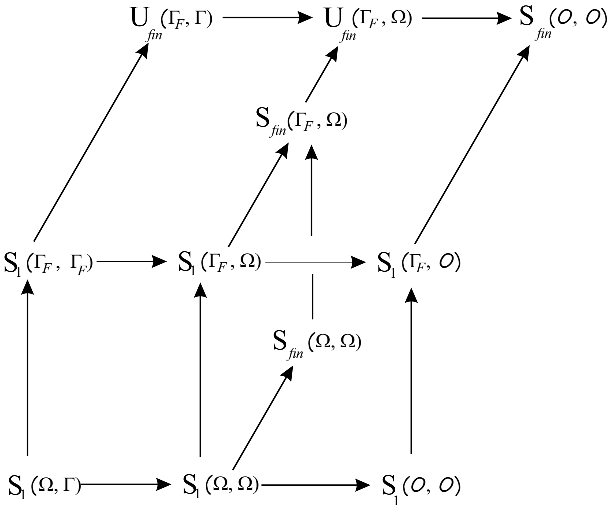
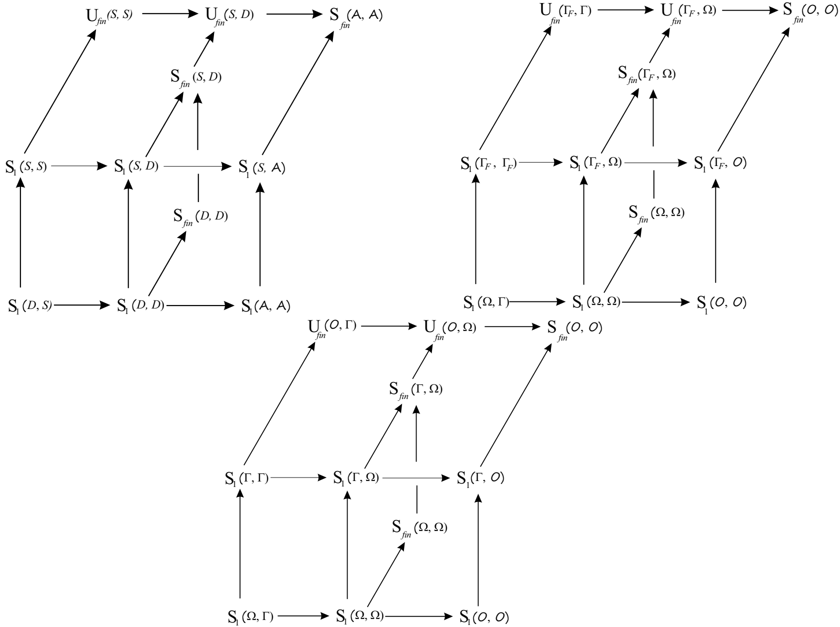


Fig. 3. The Diagram of selection principles for metrizable separable space X corresponding to selectors for sequences of dense sets of $C_p(X)$


U_fin(S, S)
U_fin(S, D)
S_fin(A, A)
S_fin(S, D)
S_1(S, S)
S_1(S, D)
S_1(S, A)
S_fin(D, D)
S_1(D, S)
S_1(D, D)
S_1(A, A)
U_fin(Γ_F, Γ)
U_fin(Γ_F, Ω)
S_fin(O, O)
S_fin(Γ_F, Ω)
S_1(Γ_F, Γ_F)
S_1(Γ_F, Ω)
S_1(Γ_F, O)
S_fin(Ω, Ω)
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S_1(Ω, Ω)
S_1(O, O)

Reference

A.V. Arhangel'skii, *Some problems and lines of investiqation in general topology*, Comment. Math. Univ. Carolinae 29, (1988), 611–629.

M. Bonanzinga, F. Cammaroto, M. Matveev, *Projective versions of selection principles*, Topology and its Applications, 157, (2010), 874–893.

W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, *The combinatorics of open covers, II*, Topology and its Applications, 73, (1996), 241–266.

Lj.D.R. Kočinac, *Selection principles and continuous images*, Cubo Math.J. 8 (2) (2006) 23–31.

M. Scheepers, *Combinatorics of open covers (I): Ramsey Theory*, Topology and its Applications, 69, (1996), p. 31–62.

Reference

B. Tsaban, *Strong γ-sets and other singular spaces*, Topology and its Applications, 153, (2005) p. 620–639.

A.V. Osipov, *Classification of selectors for sequence of dense sets of $C_p(X)$*, https://arxiv.org/abs/1708.06404