Further results on selection properties in bitopological spaces and texture structures

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Frontiers of selection principles, Warsaw, Poland

20 August-1 September 2017

Classical Selection Principles

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consist of families of subsets of an infinite set X. Then:

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

- $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ is the *Menger property*.
- $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ is the *Rothberger property*.

Selection principles in bitopological spaces

- There are a few works dealing with bispaces and selection principles.
- **[Kočinac, Özçağ, 2011]** studied selective versions of separability in bitopological spaces.
  In that paper we defined $\mathrm{M}, \mathrm{R}, \mathrm{H}$-separability and their relations in bitopological spaces. We also investigated these properties in function spaces endowed with the topology of pointwise convergence and the compact-open topology.
- **[Kočinac, Özçağ, 2012]** We began a systematic study of selection principles in bitopological context.
- **[Özçağ, 2016]** We study selective versions of separability in bitopological context by using the notions of θ-closure and θ-density and considered games associated to these properties.

M,H, R- separability in bitopological spaces

Denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the collection of all dense subsets of (X,τ_1) and (X,τ_2), respectively. $\mathcal{D}^{gp}$ is the collection of groupable dense subsets of a space; a countable dense subset D of a space Z is **groupable** if $D=\bigcup_{n\in\mathbb{N}} A_n$, each A_n finite and each open set U in Z intersects all but finitely many A_n. We say that X is:

$\mathsf{M}_{(\tau_i,\tau_j)}$**-separable** $(i,j=1,2;\ i\neq j)$, if for each sequence $<D_n : n\in\mathbb{N}>$ of elements of $\mathcal{D}_i$ there are finite sets $F_n\subset D_n$, $n\in\mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} F_n\in\mathcal{D}_j$, i.e. if $\mathsf{S}_{fin}(\mathcal{D}_i,\mathcal{D}_j)$ holds;

$\mathsf{R}_{(\tau_i,\tau_j)}$**-separable** if $\mathsf{S}_1(\mathcal{D}_i,\mathcal{D}_j)$ holds;

$\mathsf{H}_{(\tau_i,\tau_j)}$**-separable** if for each sequence $<D_n : n\in\mathbb{N}>$) of elements of $\mathcal{D}_i$ there are finite sets $F_n\subset D_n$, $n\in\mathbb{N}$, such that each τ_j-open subset of X intersects F_n for all but finitely many n;

$\mathsf{GN}_{(\tau_i,\tau_j)}$**-separable** if $\mathsf{S}_1(\mathcal{D}_i,\mathcal{D}_j^{gp})$ holds.

M,H, R- separability in bitopological spaces

Obviously,

$$\mathrm{GN}_{(\tau_i,\tau_j)} - \text{separability} \Rightarrow \mathrm{R}_{(\tau_i,\tau_j)} - \text{separability} \Rightarrow \mathrm{M}_{(\tau_i,\tau_j)} - \text{separability},$$

and

$$\mathrm{H}_{(\tau_i,\tau_j)} - \text{separability} \Rightarrow \mathrm{M}_{(\tau_i,\tau_j)} - \text{separability}.$$

- $\mathrm{R}_{(\tau_i,\tau_j)}$-separability implies separability of (X,τ_j)
  If $\tau_1 \leq \tau_2$ then:
- $\mathrm{R}_{(\tau_1,\tau_2)}$-separability implies (X,τ_1) is R-separable and (X,τ_2) is separable.
- (X,τ_1) is R-separable $\Rightarrow$ (X,τ_1,τ_2) is $\mathrm{R}_{(\tau_2,\tau_1)}$-separable $\Rightarrow$ (X,τ_1) is separable.
- If Y is either d-dense or d-open in (X,τ_1,τ_2) and X is $\mathrm{R}_{(\tau_i,\tau_j)}$-separable, then Y is also $\mathrm{R}_{(\tau_i,\tau_j)}$-separable.

Bitopological M^θ- and R^θ-separability

(X, τ_1, τ_2) bitopological space. Then X is :

- $\mathsf{M}^\theta{}_{(\tau_i,\tau_j)}$-*separable* if X satisfies $\mathsf{S}_{fin}(\mathcal{D}_i^\theta, \mathcal{D}_j^\theta)$;
- $\mathsf{R}^\theta{}_{(\tau_i,\tau_j)}$-*separable* if X satisfies $\mathsf{S}_1(\mathcal{D}_i^\theta, \mathcal{D}_j^\theta)$;
- (X, τ_1, τ_2) is $\mathsf{H}^\theta{}_{(\tau_i,\tau_j)}$-*separable* if for each sequence $< D_n : n \in \mathbb{N} >$ of elements of $\mathcal{D}_i^\theta$ there are finite sets $F_n \subset D_n$ such that the τ_j-closure of each τ_j-open set G of X intersects F_n for all but finitely many n.
- $\mathsf{GN}^\theta{}_{(\tau_i,\tau_j)}$-*separable* if $\mathsf{S}_1(\mathcal{D}_i^\theta, \mathcal{D}_j^{\theta^{(gp)}})$ holds.

Since $\mathcal{D}_i \subseteq \mathcal{D}_i^\theta$ we obtain the following implications for S_{fin}.

$\mathsf{S}_{fin}(\mathcal{D}_i^\theta, \mathcal{D}_j) \to \mathsf{S}_{fin}(\mathcal{D}_i, \mathcal{D}_j)$

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$\mathsf{S}_{fin}(\mathcal{D}_i, \mathcal{D}_j) \to \mathsf{S}_{fin}(\mathcal{D}_i, \mathcal{D}_j^\theta)$

$\mathsf{S}_{fin}(\mathcal{D}_i^\theta, \mathcal{D}_j^\theta) \to \mathsf{S}_{fin}(\mathcal{D}_i, \mathcal{D}_j^\theta)$

Games

- We investigate relations between bitopological selective versions of θ-separability and games.
- The following game is corresponded to the selection principle $\mathsf{S}_1(\mathcal{D}_i^\theta, \mathcal{D}_j^\theta)$.

$\mathsf{G}_{ij}^{pp}(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the n-th round ONE chooses a a nonempty τ_i-open set U_n, and TWO responds by choosing a point $x_n \in \mathrm{Cl_i}(U_n)$. ONE wins a play $(U_1, x_1; \cdots ; U_n, x_n; \cdots)$ if the set $\{x_n : n \in \mathbb{N}\}$ is θ-dense in (X, τ_2) otherwise, TWO wins.

Theorem

The followings are equivalent.

(1) ONE has a winning strategy in the game $\mathsf{G}_1(\mathcal{D}_i^\theta, \mathcal{D}_j^\theta)$ on X.

(2) TWO has a winning strategy in the game G_{ij}^{pp}on X

Almost (Weakly) Mengerness in bitopological spaces

Definition: [Kočinac, Özçağ, 2012] (X, τ_1, τ_2) is said to be (i, j)-*almost Menger* (weakly Menger), if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}}(\bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_j}(V))$ $(X = \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n).)$,

- (X, τ_1) almost Menger, $\tau_2 \leq \tau_1$, then (X, τ_1, τ_2) is $(1, 2)$-almost Menger.

Proposition: [Özçağ, Eysen 2016]
(X, τ_1) is Menger $\implies$ (X, τ_1, τ_2) is $(1, 2)$-almost Menger.

Example 1: X Euclidean plane, τ_1 Sorgenfrey topology, τ_2 usual topology. (X, τ_1, τ_2) is $(1, 2)$-almost Menger, but (X, τ_1) is not Menger.

Example 2: $\mathbb{R}$ real numbers, τ Euclidean topology, $\mathbb{Q}$ rational numbers. Define a topology τ', the pointed rational extension of $\mathbb{R}$, to be the topology generated by all sets $\{x\} \cup (\mathbb{Q} \cap U)$ where $x \in U \in \tau$. $(\mathbb{R}, \tau', \tau)$ is (τ', τ)-almost Menger but $(\mathbb{R}, \tau')$ is not Menger.

Theorem :

Let (X, τ_1, τ_2) be (i,j)-regular bitopological space
(X, τ_1) is Menger $\Leftrightarrow$ (X, τ_1, τ_2) is $(1,2)$-almost Menger.

Theorem :

(i,j)-almost Mengerness $\implies$ (i,j)-weakly Mengerness

- The converse of this proposition is not true in general.

Example 3: $\mathbb{R}$ real numbers with the Euclidean topology τ. For each irrational x we choose a sequence $\{x_n\}$ of rationals converging to x in τ. The rational sequence topology τ' is defined by declaring each rational open, and selecting the sets $\mathcal{U}_n(x) = \{x_i : n \leq i \leq \infty\} \cup \{x\}$ as a basis for the irrational point x. Obviously, τ' is finer than τ.

- $(\mathbb{R}, \tau', \tau)$ is not (τ', τ)-almost Menger.
- $(\mathbb{R}, \tau', \tau)$ is (τ', τ)-weakly Menger, because the set of rational numbers is dense in $(\mathbb{R}, \tau)$.

- Under which conditions both properties are equivalent?

Theorem :

If a bitopological space (X, τ_1, τ_2) is (i,j)-weakly Menger and (X, τ_j) is P-space then (X, τ_1, τ_2) is (i,j)-almost Menger.

A bitopological space (X, τ_1, τ_2) is said to be (i,j)-weakly P-space, if for every countable family $\{U_n : n \in \mathbb{N}\}$ of τ_i-open subsets of X, $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} U_n) = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n)$.

Theorem :

Let (X, τ_1, τ_2) be an (i,j)-weakly P-space. If (X, τ_1, τ_2) is (i,j)-weakly Menger, then (X, τ_1, τ_2) is (i,j)-almost Menger.

Some properties

- Let (X, τ_1, τ_2) be (i, j)-almost (weakly) Menger bitopological space and (Y, σ_1, σ_2) be a bitopological space. If $f : X \to Y$ is (i, j)-pairwise almost continuous surjection then (Y, σ_1, σ_2) is (i, j)-almost (weakly) Menger.
- Every τ_i-closed and τ_j-open subset of an (i, j)-almost(weakly) Menger bitopological space is (i, j)-almost(weakly) Menger.
- Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces and $f : X \to Y$ be a (j, i)-preopen and perfect mapping. If (Y, σ_1, σ_2) is a (i, j)-almost(weakly) Menger, then (X, τ_1, τ_2) is (i, j)-almost(weakly) Menger.
- If (X, τ_1, τ_2) is an (i, j)-almost (weakly) Menger and (Y, σ_1, σ_2) is a d-compact bitopological space, then their product $X \times Y$ is (i, j)-almost(weakly) Menger.

Almost Alster bitopological spaces

- $\mathcal{U}$ of X by G_δ-subsets is Alster cover if each compact subset of X is covered by finitely many members of $\mathcal{U}$.
- (X, τ_1, τ_2) is said to be **(i, j)- almost Alster**, if for every τ_i-Alster cover $\mathcal{U}$ of X there is a countable subset $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup_{V \in \mathcal{V}} \mathrm{Cl}_{\tau_j}(V) = X$

Theorem: (X, τ_i) metrizable.
(X, τ_1, τ_2) is (i, j)- almost σ-compact $\Leftrightarrow$ X is (i, j)- almost Alster.

- (X, τ_1, τ_2) bispace. Every τ_i-closed and τ_j-open subset of an (i, j)–almost Alster bitopological space is (i, j)-almost Alster.
- If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are (i, j)-almost Alster bitopological spaces then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is (i, j)-almost Alster.
- If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is d-continuous surjection and X is an (i, j)-almost Alster bispace, then Y is (i, j)-almost Alster.
- If (X, τ_1, τ_2) is (i, j)-almost Alster and (Y, σ_1, σ_2) is (i, j)-almost Menger then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is (i, j)-almost Menger.

- (X, τ_1, τ_2) is (i, j)- almost Lindelöf if for every τ_i-open cover $\mathcal{U}$ of X, there exists a countable $\{U_n : n \in \mathbb{N}\}$ such that $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n) = X$.
- (i, j)- almost Alster bispaces are (i, j)- almost Lindelöf.

Counterexample ???

Theorem If a bitopological space (X, τ_1, τ_2) is (i, j)- almost Lindelöf and (X, τ_i) is P-space then X is (i, j)- almost Alster.

Corollary:

Let (X, τ_1, τ_2) be a bitopological space with (X, τ_i) is P-space. The following statements are equivalent:

(1) (X, τ_i) is (i, j)-almost Alster.

(2) (X, τ_1, τ_2) is (i, j)-almost Menger;

(3) (X, τ_1, τ_2) is (i, j)-almost Lindelöf.

Texture Spaces,(Brown L.M.,1998)

Let S be a set. $\mathcal{S} \subseteq \mathcal{P}(S)$ is called **texturing** of S, if

(1) $(\mathcal{S}, \subseteq)$ is a complete lattice containing S and $\emptyset$
$\bigwedge_{j \in J} A_j = \bigcap_{j \in J} A_j, \quad A_j \in \mathcal{S}, \quad j \in J$ while
$\bigvee_{j \in J} A_j = \bigcup_{j \in J} A_j, \quad A_j \in \mathcal{S}, \quad j \in J$ for all finite index sets J.

(2) $\mathcal{S}$ is completely distributive.

(3) $\mathcal{S}$ separates the points of S. We call $(S, \mathcal{S})$ a **texture space**.

1. $(X, \mathcal{P}(X), \pi_X)$ *discrete texture* . $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complementation. $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.
2. $\mathbb{I} = [0,1]$ define $\mathcal{I} = \{[0,t] \mid t \in [0,1]\} \cup \{[0,t) \mid t \in [0,1]\}$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, $P_t = [0,t]$ and $Q_t = [0,t)$ for all $t \in I$.
3. The texture $(\mathbb{L}, \mathcal{L})$ is defined by $\mathbb{L} = (0,1]$ and $\mathcal{L} = \{(0,r] \mid r \in [0,1]\}$. For $r \in \mathbb{L}$ $P_r = (0,r] = Q_r$

Ditopology

Ditopology

A **ditopology** on $(S, \mathcal{S})$ is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

(τ_1) $S, \emptyset \in \tau$,
(τ_2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$,
(τ_3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,
(κ_1) $S, \emptyset \in \kappa$,
(κ_2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$,
(κ_3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

- $(S, \mathcal{S}, \tau, \kappa)$: **Ditopological texture space**.

Let (τ, κ) be a ditopology on$(S, \mathcal{S})$ and take $A \in \mathcal{S}$. The family $\{G_i \mid i \in I\}$ is called **open cover** of A if $G_i \in \tau$ for all $i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$. The family $\{F_i \mid i \in I\}$ is called **closed cocover** of A if $F_i \in \kappa$ for all $i \in I$ and $\bigcap_{i \in I} F_i \subseteq A$.

Dicompactness

Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$ and $A \in \mathcal{S}$.

1. A is called **compact** if whenever $\{G_i \mid i \in I\}$ is an open cover of A then there is a finite subset J of I with $A \subseteq \bigcup_{j \in J} G_j$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called compact if S is compact.
2. A is called **cocompact** if $\{F_i \mid i \in I\}$ is a closed cocover of A then there is a finite subset J of I with $\bigcap_{j \in J} F_j \subseteq A$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called cocompact if $\emptyset$ is cocompact.
3. (τ, κ) is called **stable** if every $K \in \kappa$ with $K \neq S$ is compact.
4. (τ, κ) is called **costable** if every $G \in \tau$ with $G \neq \emptyset$ is cocompact.

A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called **dicompact** if it is compact, cocompact, stable and costable.

Menger property of Texture Spaces

Let $(S, \mathcal{S})$ be a texture. A subset $\mathcal{C}$ of $\mathcal{S}$ is said to be a *cover* of a set $A \subset S$ if $A \subset \bigvee \mathcal{C}$; if $\bigvee \mathcal{C} = S$, then $\mathcal{C}$ is said to be a **cover of S**. By $\mathbb{C}$ (or $\mathbb{C}_S$ when it is necessary) we denote the family of all covers of S.

Definition

A texture space $(S, \mathcal{S})$ is said to be **Menger** if S satisfies the selection property $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$.

Theorem

For a Lindelof texture space $(S, \mathcal{S})$ the following statements are equivalent:
(1) S has the Menger property $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$.
(2) ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathbb{C}, \mathbb{C})$ on S.

Menger property of ditopological texture spaces

Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \subset S$.

A is said to have the **Menger property** if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of A there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $A \subseteq \bigvee_{n\in\mathbb{N}} \bigvee \mathcal{V}_n$.

- $(S, \mathcal{S}, \tau, \kappa)$ is Menger if the set S is Menger, $\mathsf{S}_{fin}(\theta_S, \theta_S)$
- Rothberger property $\mathsf{S}_1(\theta_S, \theta_S)$

A is said to have the **co-Menger property** if for each sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of closed cocovers of A there is a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{F}_n \subseteq \mathcal{K}_n$ and $\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n$ is a closed cocover of A.

- We say that $(S, \mathcal{S}, \tau, \kappa)$ is co-Menger if $\emptyset$ is co-Menger. $\mathsf{S}_{cfin}(\Phi_S, \Phi_S)$.

Counterexample

- There is a ditopological texture space which is Rothberger (hence Menger), but not compact.

Let $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ be the real line with the texture
$\mathcal{R} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$,
topology $\tau_{\mathbb{R}} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$
cotopology $\kappa_{\mathbb{R}} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. This ditopological texture space is neither compact (because the open cover $\mathcal{U} = \{(-\infty, n) : n \in \mathbb{N}\}$ does not contain a finite subcover) nor cocompact (because its closed cocover $\{(-\infty, n] : n \in \mathbb{N}\}$ does not contain a finite cocover). But $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is Rothberger and co-Rothberger. Let us prove that this space is Rothberger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $\mathbb{R}$. Write $\mathbb{R} = \cup\{(-\infty, n) : n \in \mathbb{N}\}$. For each n, $\mathcal{U}_n$ is an open cover of $\mathbb{R}$, hence there is some $r_n \in \mathbb{R}$ such that $(-\infty, n) \subseteq (-\infty, r_n) \in \mathcal{U}_n$. Then the collection $\{(-\infty, r_n) : n \in \mathbb{N}\}$ shows that $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is Rothberger.

Di-Menger ditopological texture spaces

- A ditopological space $(S, \mathcal{S}, \tau, \kappa)$ *M-stable* if every $K \in \kappa$ with $K \neq S$ is Menger, and *M-costable* if every $G \in \tau$ with $G \neq \emptyset$ is co-Menger.
- $(S, \mathcal{S}, \tau, \kappa)$ is **di-Menger** if it is Menger, co-Menger, M-stable and M-costable.

Theorem :

Let (f, F) be a surjective bicontinuous difunction from a di-Menger ditopological texture space $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ to ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also di-Menger.

Hurewicz-type properties of textures

$(S, \mathcal{S}, \tau, \kappa)$ $A \subset S$

A is said to have the **Hurewicz property** if for each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of A there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $A \subseteq \bigvee_{n\in\mathbb{N}} \bigcap_{m>n}(\bigcup \mathcal{V}_m)$. We say that $(S, \mathcal{S}, \tau, \kappa)$ is **Hurewicz** if the set S is Hurewicz.

Definition

A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ **di-Hurewicz** if it is Hurewicz, co-Hurewicz, H-stable and H-costable.

Theorem (Kočinac, Özçağ, 2017)

Let (f, F) be a surjective bicontinuous difunction from a di-Hurewicz ditopological texture space $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ to a ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also di-Hurewicz.

Di-uniform case

A **quasi-uniformity** on a nonempty set X is a filter $\mathcal{Q}$ on $X \times X$ satisfying

(Q1) each $U \in \mathcal{Q}$ contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$ of X;

(Q2) for each $U \in \mathcal{Q}$ there is $V \in \mathcal{Q}$ with $V \circ V \subseteq U$,

A quasi-uniformity $\mathcal{Q}$ on X is a *uniformity* if

(Q3) for each $U \in \mathcal{Q}$, $U^{-1} := \{(x,y) \in X \times X : (y,x) \in U\} \in \mathcal{Q}$.

Theorem (Özçağ, Brown, 2006)

$\mathcal{Q}$ be a quasi-uniformity on X, $\mathcal{Q}^{-1}$ its conjugate. Then the direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ corresponding to $\mathcal{Q}^{-1}$ is the complement of the direlational uniformity corresponding to $\mathcal{Q}$, that is $u(\mathcal{Q}^{-1}) = u(\mathcal{Q})'$.

Theorem (Özçağ, Brown, 2006)

$\mathcal{Q}$ be the quasi-uniformity on X, Then $\mathcal{Q}$ is a uniformity if and only if the corresponding di-uniformity $u(\mathcal{Q})$ on $(X, \mathcal{P}(X), \pi_X)$ is complemented.

Theorem,(Kočinac, Künzi, 2013)

Quasi-uniform space $(X, \mathcal{Q})$ is **pre-Menger** if for each sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{Q}$ there is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[F_n]$

1. $(S, \mathcal{S}, \mathcal{U})$ is **Menger bounded** if for each sequence $((d_n, D_n))_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ there is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of S such that $\{d_n^{\to} F_n : n \in \mathbb{N}\}$ is a cover of S, i.e. $S = \bigvee_{n\in\mathbb{N}} d_n^{\to} F_n$;

Theorem,(Kočinac, Özçağ, 2017)

Let $(X, \mathcal{Q})$ be a pre-Menger quasi-uniform space. Then the corresponding di-uniform space $(X, \mathcal{P}(X), u(\mathcal{Q}), \pi_X)$ is Menger bounded.

Theorem

If $(X, \mathcal{Q})$ is pre-Menger Lebesgue quasi-uniform space, then $(X, \mathcal{P}(X), \tau_{u(\mathcal{Q})}, \kappa_{u(\mathcal{Q})})$ is a Menger ditopological space.

Continuing works

- Let $\mathcal{U}$ be a direlational uniformity on a plain texture $(S, \mathcal{S})$. Then $(S, \mathcal{S}, \mathcal{U})$ is called totally bounded if for each $(d, D) \in \mathcal{U}$ there exists $s_1, s_2, s_3, ...s_n \in S$ for which the family $(d[s_1], D[s_1]), (d[s_2], D[s_2}), (d[s_3], D[s_3])....(d[s_n], D[s_n])$ is a dicover of $(S, \mathcal{S})$.
- A plain direlational uniform texture space $(S, \mathcal{S}, \mathcal{U})$ is uniformly di-Menger if for each sequence $((d_n, D_n))_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ there is sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of S such that the family $\{(d_n[x], D_n[x]) : x \in F_n, n \in \mathbb{N}\}$ is a dicover of $(S, \mathcal{S})$.

???

- Let $(S, \mathcal{S}, \mathcal{U})$ be a plain direlational uniform texture space. If $(S, \mathcal{S}, \mathcal{U})$ is totally bounded then it is uniformly di-Menger.

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Thank you very much...