Some results of this presentation are discussed in arXiv:1701.04322v1.

On regular but not completely regular spaces

by

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In the autumn two years ago, Taras Banakh talked about Mysior's example at the seminar in Katowice. I did not sleep (probably?) during Taras' story, because I was surprised to find that I heard the same story by Adam Mysior over thirty years ago and in the same lecture hall. Then I thought that it is like a good theatrical play. So, I would be playing it year after year – in fact two times (!) – during a lecture on the general topology for students of the fourth year of mathematics at my university.

Preparing to play leading role, I started a discussion with P. Kalemba. The first example of a regular and not completely regular space was given by A. Tychonoff (1930). So, we suspected that his method was transformed into mathematical folklore, i.e., it has been modified by various authors and omitted by the authors of the textbooks as too meticulous. Soon, we noticed that K. Ciesielski and J. Wojciechowski wrote an article about the Mysior example (November 2015) in Topology Proceedings.

Looking at the paper by Ciesielski and Wojciechowski we have noticed a declaration that *their version requires no algebraic structure on the plane, while such structure was used by Mysior*. In addition, they have organized the article as comments on the connectedness, but we hoped it is rather a topic that should examine geometric aspects of the plane.

At first, we decided to find the minimum set of properties that are necessary to describe an example of a completely regular space which has a one-point extension to a regular but not completely regular space, because Mysior's example has such property.

We obtained – rather developed a generalization – some examples, then we started to think about what constituted their diversity.

Our generalization of Mysior's example: essential notions

Let κ be an uncountable cardinal and

$$\{A(k)\colon k \in \mathbb{Z}\}$$

be a countable infinite partition of κ into pairwise disjoint subsets, each one of the cardinality κ. Let also

$$\Delta = \{(x,x)\colon x \in \kappa\} \text{ and } \Delta(k) = \Delta \cap A(k) \times A(k).$$

Fix an infinite cardinal number $\lambda < \kappa$ and proper λ^+-complete ideals $I(k,\lambda)$ on the sets $A(k)$. We assume that each $I(k,\lambda)$ contains all singletons from $A(k)$, so we get

$$\text{if } H \subset A(k) \text{ and } |H| \leqslant \lambda, \text{ then } H \in I(k,\lambda).$$

Spaces $X(\lambda, \kappa)$

We define a completely regular space $X(\lambda, \kappa)$ as follows. A topology is generated on $\kappa \times \kappa$ by the basis consisting of all single point collections $\{(a, b)\} \subset \kappa \times \kappa$, for $a \neq b$, and the sets

$$\Gamma(x, G, F) = \{(x, x)\} \cup \{x\} \times (A(k-1) \setminus G) \cup ((A(k+1) \setminus F) \times \{x\}),$$

where $x \in A(k)$ and $|G| < \lambda$ and $F \in I(k+1, \lambda)$.

Fact: *The space $X(\lambda, \kappa)$ is completely regular.*

Indeed, this space is T_1 and its basis consists of closed-open sets.

Lemma [The eEssence of the case]

Assume that $H \subseteq \Delta(k) \cap V$, where V is an open set in $X = X(\lambda, \kappa)$. If the set $\{x \in A(k) : (x, x) \in H\}$ does not belong to the ideal $I(k, \lambda)$, then the difference $\Delta(k-1) \setminus cl_X(V)$ has the cardinality less than λ.

A proof

Proof.

Suppose that a set $\{(b_\alpha, b_\alpha) : \alpha < \lambda\} \subseteq \Delta(k-1)$ of the cardinality λ is disjoint from $cl_X(V)$. For each $\alpha < \lambda$, fix a basic set $\Gamma(b_\alpha, G_\alpha, F_\alpha)$ disjoint from V, where $F_\alpha \in I(k, \lambda)$. The ideal $I(k, \lambda)$ is λ^+-complete and the set $\{x \in A(k) : (x, x) \in H\}$ does not belong to this ideal. So, there exists a point $(x, x) \in H$ such that

$$x \in A(k) \setminus \bigcup \{F_\alpha : \alpha < \lambda\}. \text{ Therefore}$$

$$(x, b_\alpha) \in (A(k) \setminus F_\alpha) \times \{b_\alpha\} \subseteq \Gamma(b_\alpha, G_\alpha, F_\alpha) \text{ for } \alpha < \lambda,$$

i.e., no (x, b_α) belongs to V. Fix a basic set $\Gamma(x, G_x, F_x) \subseteq V$ and choose $\alpha < \lambda$ such that $b_\alpha \in A(k-1) \setminus G_x$. We get $(x, b_\alpha) \in \{x\} \times (A(k-1) \setminus G_x) \subseteq V$, a contradiction. □

A few facts

Fact: Subsets $\Delta(k+1)$ and $\Delta(k)$ are closed and disjoint. By our lemma, if a set $V \subseteq X$ is open and $\Delta(k+1) \subseteq V$, then $cl_X(V) \cap \Delta(k) \neq \emptyset$ which implies that any space $X(\lambda, \kappa)$ is not normal.

Fact***:** *If $f : X(\lambda, \kappa) \to \mathbb{R}$ is a continuous real valued function, then there exists $a \in \mathbb{R}$ such that $f(x, x) = a$ for all but λ many $x \in \kappa$. Moreover, when λ has an uncountable cofinality, then $f(x, x) = a$ for all but less than λ many $x \in \kappa$.*
For a proof, which uses λ^+-completeness of the ideals $I(\lambda, \kappa)$, see Theorem 6 in arXiv: 1701.04322v1.

Fact: *Any space $X(\lambda, \kappa)$ has a one-point extension to a regular space which is not completely regular.*
Indeed, this is a standard construction (folklore?!), so we should describe it briefly.

A proof of the last fact

Proof.

Fix a point $q \notin X(\lambda, \kappa)$ and the set $X^* = X \cup \{q\}$. Then introduce the following topology: The open sets in $X(\lambda, \kappa)$ are open in X^*; and the sets

$$\mathcal{V}_m = \{q\} \cup \bigcup \{A(n) \times \kappa : n > m\}$$

form a base at the point q. For each $m \in \mathbb{Z}$, we get

$$\mathcal{V}_m \cup \Delta(m) = cl_{X^*}(\mathcal{V}_m),$$

so X^* is a regular space. Each set $\Delta(m)$ is closed in X^*. Because of **Fact***, there is no continuous function $f : X^* \to \mathbb{R}$ such that $f(q) = 1$: in fact $\Delta(m+k) \subseteq f^{-1}(1)$ for $k > 0$; and $\Delta(m) \subseteq f^{-1}(0)$. So, the space X^* is not completely regular. □

A closed subset $F \subseteq X$ is *regular closed* whenever $F = cl_X int_X(F)$, i.e., F is equal to the closure of a non-empty open set. We say that a regular space X is *rc-regular* – probably it is a property known to specialists working in the field of Boolean algebra or with different (weak) definitions of normality – whenever any two its disjoint and regular closed sets can be separated by open sets, i.e., if F and G are disjoint and regular closed, then there exist disjoint open sets V and U such that $F \subseteq V$ and $G \subseteq U$.

Theorem: If X is an rc-regular space, then any regular one-point extension of X has to be completely regular.

Proof. The standard proof of Urysohn's lemma works. □

Corollary: Spaces $X(\lambda, \kappa)$ are not rc-regular. Note that sets $\mathcal{V}_m \cup \Delta(m)$ and $X(\lambda, \kappa) \setminus \mathcal{V}_{m+1}$ are disjoint regular closed and can not be separated by open sets, for each $m \in \mathbb{Z}$. As well, each space $X(\lambda, \kappa)$ has a regular and not completely regular one-point extension, so one can apply the above theorem.

Remark: *If a regular space is an one-point extension of a normal space, then it has to be normal.*

Assume $\omega < \lambda_1 < \lambda_2 < \kappa$. We get two (non-comparable or incomparable) non-homeomorphic spaces $X(\omega, \lambda_1)$ and $X(\lambda_2, \kappa)$, since the first one has the cardinality λ_1, but every subspace of $X(\lambda_2, \kappa)$ of the cardinality λ_1 has to be discrete. In other words, spaces $(X(\omega, \lambda_1))^*$ and $(X(\lambda_2, \kappa))^*$ have non-comparable regularity ranks, which we understand as follows.

Definition of non-comparable regularity ranks: Spaces X and Y have non-comparable *regularity ranks*, whenever X and Y are regular but not completely regular and there is no regular and not completely regular space Z such that Z is homeomorphic to a subspace of X and Z is homeomorphic to a subspace of Y.

Why topological ranks appear to be inconvenient

Recall that spaces X and Y have incomparable *topological ranks*, whenever X is not homeomorphic to a subspace of Y and Y is not homeomorphic to a subspace of X. The concept of topological ranks was developed in the Polish School of Mathematics. For example, the typical results for the class $\mathcal{P}$, consisting of countable, metric and scattered spaces are as follows.

There exist continuum many non-homeomorphic spaces in $\mathcal{P}$, see S. Mazurkiewicz and W. Sierpiński (1920).

There exist ω_1 many various topological ranks between spaces from $\mathcal{P}$, a consequence of papers by S. Mazurkiewicz and W. Sierpiński (1920) and by B. Knaster and K. Urbanik (1953).

Thus, topological ranks are inconvenient for our purposes. The class of spaces of the sum form $X \oplus Y$, where X is regular and not completely regular and Y runs through (normal) spaces with different topological ranks, gives you easy answers on diversity!

On a counterexample machine

By reviewing some papers and blogs, we have noticed that some authors claim that there is a so-called Jones' counterexample machine, which works as follows. Start from a completely regular space X, which is not normal. It is needed that X contains disjoint closed subsets which, even after removal from each of them a small subset, cannot be separated by open sets. By numbering these closed sets as $\Delta_X(k)$ and assuming that the collections of small sets form proper ideals $I_X(k)$, we can check that:
If a set $V \subseteq X$ is open and the set $\Delta_X(k) \setminus V$ belongs to $I_X(k)$, then the set $\Delta_X(k-1) \setminus cl_X(V)$ belongs to $I_X(k-1)$.
Copies of X are numbered by integer and then the k-th copy is glued along the set $\Delta_X(k)$ to the $(k-1)$-copy, moreover copies of sets $\Delta_X(m)$, for $k \neq n \neq k-1$, are removed from the k-th copy. As a result we get the completely regular space $\mathbf{Y}_X$, which has a one-point extension to the regular space which is not completely regular, i.e., it is not rc-regular.

The Niemytzki plane

Recall that the Niemytzki plane $\mathbb{P} = \{(a, b) \in \mathbb{R} \times \mathbb{R} : 0 \leqslant b\}$ is endowed with the topology generated by open discs disjoint with the real axis and all sets of the form $\{a\} \cup D$ where $D \subseteq \mathbb{P}$ is an open disc which is tangent to $\Delta_{\mathbb{P}}$ at the point $a \in \Delta_{\mathbb{P}}$. Choose pairwise disjoint subsets $\Delta_{\mathbb{P}}(k) \subseteq \Delta_{\mathbb{P}}$, where $k \in \mathbb{Z}$, such that each set $\Delta_{\mathbb{P}}(k)$ meets every dense G_δ subset of the real axis.

Proposition

Let a set $F \subseteq \Delta_{\mathbb{P}}$ be a dense subset in the natural topology on the real axis $\Delta_{\mathbb{P}}$. If a set V is open in $\mathbb{P}$ and $F \subseteq V$, then the set $\Delta_{\mathbb{P}} \setminus cl_{\mathbb{P}}(V)$ is of first category in $\Delta_{\mathbb{P}}$. □

Using Jones' counterexample machine – which is (fully) analogous to the above-described generalization of the Mysior example – we get a completely regular space, which has a one-point extension to a regular and not completely regular space.

The Sorgenfrey plane

The Sorgenfrey plane $\mathbb{S} = \{(a, b) : a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$ is endowed with the topology generated by rectangles of the form $[a, b) \times [c, d)$. Let $\Delta_{\mathbb{S}} = \{(x, -x) : x \in \mathbb{R}\}$ and consider this line with the metric topology. Then choose pairwise disjoint subsets $\Delta_{\mathbb{S}}(k) \subseteq \Delta_{\mathbb{S}}$ such that each set $\Delta_{\mathbb{S}}(k)$ meets every dense G_δ subset of $\Delta_{\mathbb{S}}$.

Proposition

Let a set $F \subseteq \Delta_{\mathbb{S}}$ be a dense subset in the topology on $\Delta_{\mathbb{S}}$ which is inherited from the Euclidean topology. If a set V is open in $\mathbb{S}$ and $F \subseteq V$, then the set $\Delta_{\mathbb{S}} \setminus cl_{\mathbb{S}}(V)$ is of first category in $\Delta_{\mathbb{S}}$. □

Again, using Jones' counterexample machine, we get a completely regular space, which has a one-point extension to a regular and not completely regular space.

W. Bielas' observation

The above propositions one can prove by the second category argument, i.e., an argument which some authors use to prove that $\mathbb{S}$ and $\mathbb{P}$ are not normal.

A remark by W. Bielas: *A regular one-point extension of the Niemytzki plane has to be completely regular.*

Wojtek Bielas argued, referring to some results by D. Chodounský (2007). These results can be corrected to the following property.

A (new?) property of the Niemytzky plane

Any two regular closed and disjoint subsets of the Niemytzki plane can be separated by open sets, , i.e., the Niemytzki plane is rc-regular.

On problems

The subject I have talked about is designed in such a way that any math student (novice researcher) could easily ask questions that haven't got an immediate (or well-known) answer. I know of no textbook which goes through the details of such questions. For example, in R. Engelking's textbook the proof of hereditary normality of the Aleksandroff double circle is left to the reader. Likewise, similar problems one can attribute to the inheritance of being rc-regular!

In fact, I do not know much about topological ranks of the examples considered. This also applies to spaces, which are the union of two discrete subspace (the Mysior example has this property).

I believe that examining diversity of counterexamples is and will be an interesting research direction - not whimsical, although the results will take the form of statements about the impossibility - for every researcher.

Thank you for your attention

Postscriptum

A family of sets is called *almost disjoint*, whenever any its two members have the finite intersection. A set C *separates* two families, whenever each member of the first family is almost contained in C, i.e., $B \setminus C$ is finite for any $B \in Q$, and each member of the other family is almost disjoint with C. An uncountable family $\mathcal{L}$, which consists of almost disjoint and infinite subsets of ω, is called *Lusin-gap,* whenever no two its uncountable and disjoint subfamilies can be separated by a subset of ω. Adapting concepts of so called Mrowka spaces (or Isbell-Mrowka spaces) to a Lusin-gap $\mathcal{L}$, let a topology on $\Psi(\mathcal{L}) = \mathcal{L} \cup \omega$ be such that any subset of ω is open, and also for each point $A \in \mathcal{L}$ the sets $\{A\} \cup A \setminus F$, where F is finite, is open. The space $\Psi(\mathcal{L})$ is completely regular, its uncountable and disjoint subsets are not separable by open set, Jones' counterexample machine can be applied for it. Consistently, $\Psi(\mathcal{L})$ can not be embedded into the Niemytzki (or Sorgenfrey) plane. Does the same hold in ZFC?