On generalised Lusin sets with respect to two ideals

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Frontiers of Selection Principles
1-st September 2017, Warsaw

Definition (Cardinal coefficients)

For any $I \subset \mathscr{P}(X)$ let

$$non(I) = min\{|A| : \ A \subset X \wedge A \notin I\}$$

$$cov(I) = min\{|\mathscr{A}| : \ \mathscr{A} \subset I \wedge \bigcup \mathscr{A} = X\}$$

$$cof(I) = min\{|\mathscr{A}| : \ \mathscr{A} \subset I \wedge \mathscr{A} - \text{Borel base of } I\}$$

$\mathbb{K}$ - σ ideal of meager sets
$\mathbb{L}$ - σ ideal of null sets

Definition

Let $I, J \subset \mathscr{P}(X)$ are σ - ideals on Polish space X, I has Borel base. We say that $L \subset X$ is a (I, J) - Luzin set if

- $L \notin I$
- $(\forall B \in I)\ B \cap L \in J$

If in addition the set L has cardinality κ then L is (κ, I, J) - Luzin set.

Definition

An ideals I and J are orthogonal in Polish space X if

$$\exists A \in \mathscr{P}(X)\ A \in I \wedge A^c \in J$$

and then we write $I \perp J$.

Fact

Assume that $I \perp J$.

1. *There exist a (I, J) - Luzin set.*
2. *If L is a (I, J) - Luzin set then L is not (J, I) - Luzin set.*

If $\mathbb{R} = M \cup N$ is Marczewski decomposition then N is $(\mathbb{K}, \mathbb{L})$-Lusin set which has Baire property and is measurable.

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Definition (Tall σ-ideal)

We say that I is tall σ-ideal on Polish space when

- has Borel base,
- For any $B \in Bor \setminus I$ there is $P \in Perf \cap I$ such that $P \subseteq B$.

Definition

We say that σ-ideal J is perfectly small if any perfect set is not member of J.

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Lemma

If I, J are σ-ideals on Polish space X such that

1. *I is tall ideal,*
2. *J is perfectly small,*

then every (I, J)-Lusin set is not I measurable set in X.

Proof.

Let A be I measurable (I, J)-Lusin set. Then for some $B \in Bor \setminus I$ $B \subseteq A$.

Find $P \in Perf \cap I$ such that $P \subseteq B$.

$$P = P \cap A \in J$$

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Menger set

Let X be a Polish space and $A \subseteq X$.
We say that A is Menger set in X iff for every sequence $(\mathscr{U}_n)_{n\in\omega}$ of open covers of A there is $(\mathcal{F}_n)_{n\in\omega}$ such that

- $\mathcal{F} \in [\mathscr{U}_n]^{<\omega}$, for each $n \in \omega$,
- $\bigcup_{n\in\omega} \mathcal{F}_n$ is open cover of A.

Theorem

Let X be a Polish space then every $(\mathbb{K}, [X]^{\mathfrak{d}})$-Lusin set is Menger.

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Proof.

Let A is $(\mathbb{K}, [X]^{\mathfrak{d}})$-Lusin set and $D \in [A]^\omega$ is dense in A and $D = \{r_n : n \in \omega\}$.

Consider arbitrary $(\mathscr{U}_n)_{n\in\omega}$ of open covers of A

Find $(U_n)_{n\in\omega}$ such that $r_n \in U_n \in \mathscr{U}_n$ for every $n \in \omega$.

$$A \setminus \bigcup_{n\in\omega} U_n \subseteq \bar{A} \setminus \bigcup_{n\in\omega} U_n \in \mathbb{K}$$

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Then $A \setminus \bigcup_{n\in\omega} U_n$ is Menger.

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Theorem (Bukovsky)

If κ is uncountable regular cardinal and there are $(\kappa, \mathbb{K}, [\mathbb{R}]^{<\kappa})$ and $(\lambda, \mathbb{L}, [\mathbb{R}]^{<\lambda})$ - Luzin sets then

$$\kappa = cov(\mathbb{K}) = non(\mathbb{K}) = non(\mathbb{L}) = cov(\mathbb{L}) = \lambda.$$

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Definition

Let $\mathscr{F} \subset X^X$ be any family of functions on the Polish space X. We say that $A, B \subset X$ are equivalent with respect to $\mathscr{F}$ if

$$(\exists f, g \in \mathscr{F})\, (B = f[A] \wedge A = g[B])$$

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Theorem

Assume that X be a Polish space I, J are σ-ideals with Borel base. Let $\kappa = cov(I) = cof(I) \leq non(J)$. Let $\mathcal{F}$ be a family of functions from X to X. Assume that $|\mathcal{F}| \leq \kappa$. Then we can find a sequence $(L_\alpha)_{\alpha<\kappa}$ such that

1. *L_α is (κ, I, J) - Luzin set,*
2. *for $\alpha \neq \beta$, L_α is not equivalent to L_β with respect to the family $\mathcal{F}$.*

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Corollary

If $2^\omega = cov(I) = non(J)$ then there exists continuum many different (I, J) - Luzin sets which aren't equivalent with respect to all I-measurable functions.

Definition

We say that σ - ideal I has Fubini property iff for every Borel set $A \subset X \times X \quad \{x \in X : A_x \notin I\} \in I \implies \{y \in X : A^y \notin I\} \in I$

Lemma (folklore)

Let I be σ - ideal on 2^ω with conditions:

- $\mathbb{P}_I = Bor(2^\omega) \setminus I$ *be a proper,*
- I *has Fubini property.*

Assume that $B \in Bor(2^\omega) \cap I$ be a Borel set in $V[G]$. Then there exists $D \in V$ s.t.

$$B \cap (2^\omega)^V \subset D \in I.$$

For Cohen and Solovay reals, see Solovay, Cichoń and Pawlikowski, see [2, 4, 8]

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Definition

Let $M \subseteq N$ be standard transitive models of ZF.
Coding Borel sets from the ideal I is absolute iff

$$(\forall x \in M \cap \omega^\omega) M \models \#x \in I \leftrightarrow N \models \#x \in I.$$

Theorem

Let $\omega < \kappa$ and I, J be σ - ideals with Borel base on 2^ω,

- *$\mathbb{P}_I = Bor(2^\omega) \setminus I$ be a proper forcing notion,*
- *I has Fubini property,*
- *Borel codes for sets from ideal J are absolute.*

Then $\mathbb{P}_I = Bor(2^\omega) \setminus I$ - is preserving (I, J) - Luzin set porperty.

Proof

Let G is $\mathbb{P}_I$ generic over V and L - (κ, I, J) - Luzin set in the ground model V.
In $V[G]$ take any $B \in I$ then $L \cap B \cap V = L \cap B$
By Lemma we can find $b \in 2^\omega \cap V$ - Borel code s.t.
$B \cap V \subset \#b \in I \cap V$
But L is (I, J)-Luzin set then $L \cap \#b \in J \cap V$,
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Finally we have in $V[G]$

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Theorem

Let $(\mathbb{P}, \leq)$ *be a forcing notion such that*

$$\{B : B \in I \cap \mathrm{Borel}(\mathscr{X}), B \text{ is coded in } V\}$$

is a base for I *in* $V^{\mathbb{P}}[G]$*. Assume that Borel codes for sets from ideals* I, J *are absolute. Then* $(\mathbb{P}, \leq)$ *preserve being* (I, J) *- Luzin sets.*

Corollary

Let $(\mathbb{P}, \leq)$ be any forcing notion which does not change the reals i. e. $(\omega^\omega)^V = (\omega^\omega)^{V^{\mathbb{P}}[G]}$. Assume that Borel codes for sets from ideals I, J are absolute. Then $(\mathbb{P}, \leq)$ preserve being (I, J) - Luzin sets.

Corollary

Assume that $(\mathbb{P}, \leq)$ is a σ-closed forcing and Borel codes for sets from ideals I, J are absolute. Then $(\mathbb{P}, \leq)$ preserve (I, J) - Luzin sets.

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Measure case

Let Ω is a family of clopen sets of Cantor space 2^ω and

$$C^{random} = \{f \in \Omega^\omega : (\forall n \in \omega)\mu(f(n)) < 2^{-n}\}$$

Let us define $\sqsubseteq = \bigcup_{n\in\omega} \sqsubseteq_n$ where

$$(\forall f \in C^{random})(\forall g \in 2^\omega)(f \sqsubseteq_n g \leftrightarrow (\forall k \geq n)\ g \notin f(k)).$$

g covers N if for any $f \in C^{random} \cap N$ $f \sqsubseteq g$. We write $N \sqsupseteq g$.

Definition (almost preserving)

We say that forcing notion P almost preserving relation $\sqsubseteq^{random}$ if for any countable large enough elementary submodel $N \prec H_\kappa$ (for large enough κ)
If $N \sqsubseteq g$ and $p \in P \cap N$ then there exists stronger condition $q \in P$ which is (N, P) generic s.t. $q \Vdash$ "$N[G] \sqsubseteq g$".

Definition of the notion of preservation of relation $\sqsubseteq^{random}$ by forcing notion $(\mathbb{P}, \leq)$ can be found in paper [5]. Let us focus on the following consequence of that definition.

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Theorem (Goldstern)

If $(\mathbb{P}, \leq)$ *preserves* $\sqsubseteq^{random}$ *then* $\mathbb{P} \Vdash \mu^*(2^\omega \cap V) = 1$.

Now we say that forcing notion $\mathbb{P}$ is preserving outer measure iff $\mathbb{P}$ preserve $\sqsubseteq^{random}$.

Theorem (Goldstern,Judah, Shelah)

Random forcing and Laver forcing preserves outer measure.

Theorem (Goldstern)

Let $\mathbb{P}_\lambda = ((P_\alpha, Q_\alpha) : \alpha < \gamma)$ *be any countable support iteration such that*

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Thank You for your attention

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