Selection principles and topological games
Day 1 – General facts, duality, (un)determinacy

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- In all of the games we will consider in this tutorial, there will be two players, One and Two, playing against each other.

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$$\langle x_0, U_0, x_1, U_1, x_2, U_2, \dots, x_n, U_n, \dots \rangle,$$

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the winner is $\textsc{One}$ if $X \subseteq \bigcup_{n \in \omega} U_n$, and $\textsc{Two}$ otherwise.

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Warming up with some examples

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  Here TWO can be sure to win:
  All TWO has to do is play, in each inning $n \in \omega$, an open interval of length $\frac{1}{2^n}$.
  At the end of the play, the open set $\bigcup_{n\in\omega} U_n$ will be a union of intervals whose lengths add up to

  $$\sum_{n\in\omega} \frac{1}{2^n} = 2,$$

  hence cannot cover all of the real line.

A closer look at the previous example shows that, more generally, we have the following:

Fact

If $X \subseteq \mathbb{R}$ is not a (Lebesgue) measure zero set, then $\textsc{Two}$ has a winning strategy in the point-open game in X.

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- A strategy $\varphi : \tau^{<\omega} \to X$ for $\textsc{One}$ in the point-open game on X is a **winning strategy** for $\textsc{One}$ if, for every sequence $(U_n)_{n\in\omega}$ of open subsets of $\mathbb{R}$ such that $\forall n \in \omega\ (\underbrace{\varphi(\langle U_0, \ldots, U_{n-1}\rangle)}_{=x_n} \in U_n)$, we have

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$$X \subseteq \bigcup_{n\in\omega} U_n.$$
- A strategy $\psi : X^{<\omega} \to \tau$ for TWO in the point-open game on X is a **winning strategy** for TWO if, for every sequence $(x_n)_{n\in\omega}$ of points of X, we have $X \nsubseteq \bigcup_{n\in\omega} \underbrace{\psi(\langle x_0, \ldots, x_n\rangle)}_{=U_n}$.

Determined games

Notation

If $\textsc{Player}$ is a player of a game G, we denote by

$$\textsc{Player} \uparrow \mathsf{G}$$

the fact that $\textsc{Player}$ has a winning strategy in G,
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Determined games

Notation

If PLAYER is a player of a game G, we denote by

$$\text{PLAYER} \uparrow \mathsf{G}$$

the fact that PLAYER has a winning strategy in G,
and by

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the fact that PLAYER does not have a winning strategy in G.

Definition

A game G is

- **determined** if either $\text{ONE} \uparrow \mathsf{G}$ or $\text{TWO} \uparrow \mathsf{G}$;
- **undetermined** otherwise – i.e. if $\text{ONE} \not\uparrow \mathsf{G}$ and $\text{TWO} \not\uparrow \mathsf{G}$.

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Parenthetical remark

If G is a game of length N (for some $N \in \omega$) that allows no draws, then $\text{One} \uparrow \mathrm{G}$ means that

$$\exists a_1 \forall b_1 \exists a_2 \forall b_2 \dots \exists a_N \forall b_N \ (\text{One wins the play } \langle a_1, b_1, \dots, a_N, b_N \rangle),$$

so $\text{One} \not\uparrow \mathrm{G}$ means that

$$\forall a_1 \exists b_1 \forall a_2 \exists b_2 \dots \forall a_N \exists b_N \ (\text{Two wins the play } \langle a_1, b_1, \dots, a_N, b_N \rangle),$$

i.e. $\text{Two} \uparrow \mathrm{G}$.

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Wait. Can it be the case that none of the players have a winning strategy?
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i.e. $\text{TWO} \uparrow \text{G}$.

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(We will talk about that later.)

The Rothberger game

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- In each inning $n \in \omega$, $\textsc{One}$ chooses an open cover $\mathcal{U}_n$ of X, and then $\textsc{Two}$ picks an open set $U_n \in \mathcal{U}_n$.
- At the end of the play

$$\langle \mathcal{U}_0, U_0, \mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \ldots, \mathcal{U}_n, U_n, \ldots \rangle,$$

the winner is $\textsc{Two}$ if $X \subseteq \bigcup_{n \in \omega} U_n$, and $\textsc{One}$ otherwise.

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$$X \text{ is countable}$$
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Duality of games

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Two games G and G' are **dual** if

- $\text{One} \uparrow G \Leftrightarrow \text{Two} \uparrow G'$

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Theorem (Galvin 1978)

The point-open game and the Rothberger game are dual on every topological space.

(Note that the point-open game can be played not only on subsets of $\mathbb{R}$, but in fact on any topological space.)

Proof. ($\text{ONE} \uparrow$ point-open$(X) \Rightarrow \text{TWO} \uparrow$ Rothberger(X))

Let $\varphi : \tau^{<\omega} \to X$ be a winning strategy for ONE in point-open(X).
We will use φ to describe a winning strategy for TWO in Rothberger(X):

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Let $\mathcal{U}_0$ be One's initial move in Rothberger(X). Two will respond by choosing $U_0 \in \mathcal{U}_0$ such that $\underbrace{\varphi(\langle\rangle)}_{=x_0} \in U_0$.

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If ONE's next move is $\mathcal{U}_1$, let TWO's move be $U_1 \in \mathcal{U}_1$ with $\underbrace{\varphi(\langle U_0 \rangle)}_{=x_1} \in U_1$.

Proof. (ONE $\uparrow$ point-open(X) $\Rightarrow$ TWO $\uparrow$ Rothberger(X))

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If ONE plays $\mathcal{U}_2$ in the next inning, TWO will pick $U_2 \in \mathcal{U}_2$ with $\underbrace{\varphi(\langle U_0, U_1 \rangle)}_{=x_2} \in U_2$, and so forth.

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In each inning $n \in \omega$, TWO will choose some element U_n of ONE's move $\mathcal{U}_n$ that covers the point $\underbrace{\varphi(\langle U_0, \ldots, U_{n-1} \rangle)}_{=x_n}$.

Proof. ($\text{ONE} \uparrow \text{point-open}(X) \Rightarrow \text{TWO} \uparrow \text{Rothberger}(X)$)

By proceeding in this fashion, the sets $\langle U_0, U_1, \ldots, U_n, \ldots \rangle$ played by TWO in this play of Rothberger(X) are exactly those played by TWO in the following play of point-open(X):

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But the above is a play of point-open(X) in which ONE has made use of the winning strategy φ, hence $\bigcup_{n \in \mathbb{N}} U_n = X$

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By proceeding in this fashion, the sets $\langle U_0, U_1, \ldots, U_n, \ldots \rangle$ played by TWO in this play of Rothberger(X) are exactly those played by TWO in the following play of point-open(X):

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But the above is a play of point-open(X) in which ONE has made use of the winning strategy φ, hence $\bigcup_{n \in \mathbb{N}} U_n = X$ – which means that

$$\langle \mathcal{U}_0, U_0, \mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \ldots, \mathcal{U}_n, U_n, \ldots \rangle$$

is a play of Rothberger(X) in which TWO is the winner.

Duality of games

The point-open game and the Rothberger game are dual

Proof. (TWO $\uparrow$ Rothberger(X) $\Rightarrow$ ONE $\uparrow$ point-open(X))

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Let $\psi : \mathcal{O}^{<\omega} \to \tau$ be a winning strategy for TWO in Rothberger(X). We will define a winning strategy for ONE in point-open(X).

Proof. (Two $\uparrow$ Rothberger(X) $\Rightarrow$ One $\uparrow$ point-open(X))

Let $\psi : \mathcal{O}^{<\omega} \to \tau$ be a winning strategy for Two in Rothberger(X). We will define a winning strategy for One in point-open(X).

Lemma 0

There is $x_0 \in X$ such that, for every open neighbourhood U of x_0, there is $\mathcal{U} \in \mathcal{O}$ such that $U = \psi(\langle \mathcal{U} \rangle)$.

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Proof of Lemma 0.

Suppose that every $x \in X$ has an open neighbourhood U_x that is not of the form $\psi(\langle \mathcal{U} \rangle)$ for $\mathcal{U} \in \mathcal{O}$.

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There is $x_1 \in X$ such that, for every open neighbourhood U of x_1, there is $\mathcal{U} \in \mathcal{O}$ such that $U = \psi(\langle \mathcal{U}_0, \mathcal{U} \rangle)$.

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Proof of Lemma 1.

Pretty much the same thing we did in Lemma 0. □

Proof. (Two $\uparrow$ Rothberger(X) $\Rightarrow$ One $\uparrow$ point-open(X))

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Now let One play $x_1 \in X$ in the next inning of point-open(X).

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Now let ONE play $x_1 \in X$ in the next inning of point-open(X). TWO will respond with an open set $U_1 \subseteq X$ such that $x_1 \in U_1$.

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Now let One play $x_1 \in X$ in the next inning of point-open(X). Two will respond with an open set $U_1 \subseteq X$ such that $x_1 \in U_1$. By Lemma 1, $U_0 = \psi(\langle \mathcal{U}_0, \mathcal{U}_1 \rangle)$ for some $\mathcal{U}_1 \in \mathcal{O}$.

Proof. (TWO $\uparrow$ Rothberger$(X) \Rightarrow$ ONE $\uparrow$ point-open(X))

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Lemma 2

There is $x_2 \in X$ such that, for every open neighbourhood U of x_2, there is $\mathcal{U} \in \mathcal{O}$ such that $U = \psi(\langle \mathcal{U}_0, \mathcal{U}_1, \mathcal{U} \rangle)$.

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Big Lemma

For each $\langle \mathcal{U}_0, \ldots, \mathcal{U}_{n-1} \rangle \in \mathcal{O}^{<\omega}$, there is $x \in X$ such that, for every open neighbourhood U of x, there is $\mathcal{U} \in \mathcal{O}$ such that $U = \psi(\langle \mathcal{U}_0, \ldots, \mathcal{U}_{n-1}, \mathcal{U} \rangle)$.

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Proof of the Big Lemma.

Analogous to the previous ones. □

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By successively applying the Big Lemma, we simultaneously obtain:

Proof. (TWO $\uparrow$ Rothberger(X) $\Rightarrow$ ONE $\uparrow$ point-open(X))

- a play of point-open(X)

$$\langle x_0, U_0, x_1, U_1, \ldots, x_n, U_n, \ldots \rangle$$

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$$\langle \mathcal{U}_0, \underbrace{\psi(\langle \mathcal{U}_0 \rangle)}_{=U_0}, \mathcal{U}_1, \underbrace{\psi(\langle \mathcal{U}_0, \mathcal{U}_1 \rangle)}_{=U_1}, \ldots, \mathcal{U}_n, \underbrace{\psi(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)}_{=U_n}, \ldots \rangle.$$

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Since ψ is a winning strategy for Two in Rothberger(X), it follows that

$$\bigcup_{n \in \mathbb{N}} U_n = X$$

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– which means that ONE wins the play of point-open(X) above.

Proof.

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- (... OK, also if X is the Cantor ternary set)

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And then:

Theorem (Telgársky 1975, Galvin 1978)

Let X be a topological space in which every point is a G_δ (that is: for each $x \in X$, the set $\{x\}$ is the intersection of a countable family of open sets). Then $\textsc{Two} \uparrow$ Rothberger(X) if and only if X is countable.

Determinacy of the point-open/Rothberger game

Corollary (Pawlikowski 1994)

Let $X \subseteq \mathbb{R}$. Then Rothberger(X) is undetermined if and only if X is an uncountable Rothberger space.

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The Continuum Hypothesis implies the existence of a Luzin set (i.e. an uncountable $X \subseteq \mathbb{R}$ that has countable intersection with every nowhere dense subset of $\mathbb{R}$).

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Corollary (Pawlikowski 1994)

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The statement

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Theorem (Laver 1976)

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Corollary

It is consistent with ZFC that Rothberger(X) is determined for every $X \subseteq \mathbb{R}$.

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Is Rothberger(X) determined for every $X \subseteq \mathbb{R}$?

Answer

The statement

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is independent of ZFC!

The Menger game

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OK! How about the following?

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The **Menger game** on a topological space (X, τ) is played according to the following rules:

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Exercise

Define *(winning) strategy in the Menger game* – both for ONE and TWO.

The Menger game

Does that make that big a difference?

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In particular, $\text{Two} \uparrow \text{Menger}(\mathbb{R})$.

The Menger game

Fact

X is σ-compact

$\Downarrow$

$\textsc{Two} \uparrow$ Menger(X)

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Fact

$$X \text{ is } \sigma\text{-compact}$$
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But there's more!

The Menger game

Theorem (Telgársky 1984)

Let X be a metrizable topological space. Then $\text{Two} \uparrow \mathsf{Menger}(X)$ if and only if X is σ-compact.

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Let X be a metrizable topological space. Then $\text{Two} \uparrow \text{Menger}(X)$ if and only if X is σ-compact.

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Let X be a topological space. Then X is Menger if and only if $\text{One} \not\uparrow \text{Menger}(X)$.

So the question

Question

Is there $X \subseteq \mathbb{R}$ such that $\text{Menger}(X)$ is undetermined?

boils down to

Question (Menger 1924, Hurewicz 1926)

Is there $X \subseteq \mathbb{R}$ that is Menger but not σ-compact?

Determinacy of the Menger game

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YES.

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YES. ($\leftarrow$ And that's a ZFC result!)

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Example

If $\mathcal{O}$ is the set of all the open covers of a topological space X, then $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ is the Rothberger game on X.

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But now we have a whole new world of possibilities of games to be considered! (Which will be our main interest in this tutorial, by the way.)

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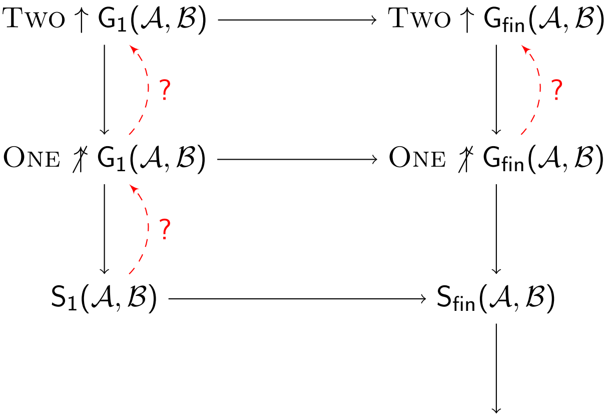

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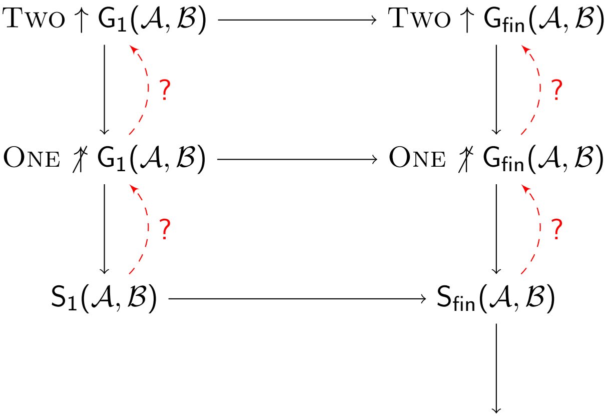

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The Rothberger game and the point-open game are dual.

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Recall

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The Rothberger game and the point-open game are dual.

In view of this result, it seems natural to seek for a game G (that should resemble the point-open game somehow) such that

the Menger game and G are dual

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A natural first candidate

The obvious thing to try seems to be the following:

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Exercise

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$\ddot\frown$

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(c) if X is T_3, then $\text{Two} \uparrow \text{Menger}(X) \Rightarrow \text{One} \uparrow \text{compact-open}(X)$.

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Are the Menger game and the compact-open game dual on every T_3 space?

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Question (Telgársky 1984)

Are the Menger game and the compact-open game dual on every T_3 space?
In other words: is it true that
$\text{TWO} \uparrow \text{compact-open}(X) \Rightarrow \text{ONE} \uparrow \text{Menger}(X)$
for every T_3 space X?

TWO $\uparrow$ *compact-open* $\overset{?}{\Rightarrow}$ ONE $\uparrow$ *Menger*

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An open cover $\mathcal{U}$ of a topological space X is a k-**cover** if for every compact $K \subseteq X$ there is $U \in \mathcal{U}$ with $K \subseteq U$.

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Analogous to point-open vs Rothberger. □

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So the question becomes:

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Does $\text{One} \uparrow \mathsf{G}_1(\mathcal{K}, \mathcal{O})$ imply $\text{One} \uparrow$ Menger if X is T_3?

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Let X be a T_3 Menger space. Must it be the case that $\text{One} \not\uparrow \mathsf{G}_1(\mathcal{K}, \mathcal{O})$?

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Answer

At least consistently, NO.

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Example (Aurichi–Dias 2014)

If there is a Sierpiński set (i.e. an uncountable $S \subseteq \mathbb{R}$ whose intersection with every measure zero set is countable), then there is a Menger space that does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

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Theorem (Sierpiński 1924)

The Continuum Hypothesis implies the existence of a Sierpiński set.

TWO $\uparrow$ *compact-open* $\not\Rightarrow$ ONE $\uparrow$ *Menger (consistently)*

Corollary

The Continuum Hypothesis implies the existence of a Menger space that does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

$\text{Two} \uparrow$ *compact-open* $\not\Rightarrow$ $\text{One} \uparrow$ *Menger (consistently)*

Corollary

The Continuum Hypothesis implies the existence of a Menger space that does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.
(There is yet another example of such a space assuming $\text{cov}(\mathcal{M}) < \mathfrak{d}$.)

TWO $\uparrow$ *compact-open* $\not\Rightarrow$ ONE $\uparrow$ *Menger (consistently)*

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Corollary (Aurichi–Dias 2014)

It is consistent with ZFC that the compact-open game and the Menger game are not dual.

$\textsc{Two} \uparrow$ *compact-open* $\not\Rightarrow$ $\textsc{One} \uparrow$ *Menger (consistently)*

Corollary

The Continuum Hypothesis implies the existence of a Menger space that does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.
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Corollary (Aurichi–Dias 2014)

It is consistent with ZFC that the compact-open game and the Menger game are not dual.

Problem

Is there a ZFC example of a topological space on which the compact-open game and the Menger game are not dual?

References

- L. F. Aurichi and R. R. Dias, *Topological games and Alster spaces*, Canadian Mathematical Bulletin **57**:4 (2014), 683–696.
- D. H. Fremlin and A. W. Miller, *On some properties of Hurewicz, Menger, and Rothberger*, Fundamenta Mathematicae **129** (1988), 17–33.
- F. Galvin, *Indeterminacy of point-open games*, Bulletin de l'Académie Polonaise des Sciences **26** (1978), 445–449.
- W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Mathematische Zeitschrift **24** (1926), 401–421.
- R. Laver, *On the consistency of Borel's conjecture*, Acta Mathematica **137** (1976), 151–169.

References

- N. Lusin, *Sur un problème de M. Baire*, Comptes Rendus de l'Académie des Sciences de Paris **158** (1914), 1258–1261.
- P. Mahlo, *Über Teilmengen des Kontinuums von dessen Machtigkeit*, Sitzungsberichte des Sachsischen Akademie der Wissenschaften zu Leipzig **65** (1913), 283–315.
- K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte der Wiener Akademie **133** (1924), 421–444.
- J. Pawlikowski, *Undetermined sets of point-open games*, Fundamenta Mathematicae **144** (1994), 279–285.
- F. Rothberger, *Eine Verschärfung der Eigenschaft* **C**, Fundamenta Mathematicae **30** (1938), 50–55.

References

- F. Rothberger, *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété* **C**, Proceedings of the Cambridge Philosophical Society **37** (1941), 109–126.
- M. Scheepers, *Combinatorics of open covers (III): games, $C_p(X)$*, Fundamenta Mathematicae **152** (1997), 231–254.
- W. Sierpiński, *Sur l'hypothèse du continu ($2^{\aleph_0} = \aleph_1$)*, Fundamenta Mathematicae **5** (1924), 177–187.
- R. Telgársky, *Spaces defined by topological games*, Fundamenta Mathematicae **88** (1975), 193–223.
- R. Telgársky, *On games of Topsøe*, Mathematica Scandinavica **54** (1984), 170–176.