Rothberger, Menger and inbetween

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- We will denote by $Suc(p) = \{q \in T : q \text{ is a successor of } p\}$.

If $p \in T$ then an open neighborhood for p is of the form $V_p \setminus \bigcup_{q \in F} V_q$, where $F \subset Suc(p)$ is finite.

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- It is Hausdorff;
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- If S is a Suslin tree, then every point has a countable local base;
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- In S, the union of the first α levels is closed for every α.

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Proposition

If $D \subset S$ is discrete, then $\overline{D}$ is countable.

Proof.

By the previous result, D is countable. Therefore it is a subset of the first α levels of S for some $\alpha < \omega_1$. Note that this union is closed and countable. □

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Lemma (Shapirovsky)

Let X be a topological space and let $(V_x)_{x \in X}$ be an ona. Then there is a discrete subset $D \subset X$ such that $X = \bigcup_{d \in D} V_d \cup \overline{D}$.

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Corollary

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Given a basic open set $V_p \setminus \bigcup_{q\in F} V_q$, we will say that $\{p\} \cup F$ is the support of this open set. Note that the support is always finite and, since we have countably many open coverings and each one is also countable, the union of all the supports considered here is also countable.

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$$r \in V_p \setminus \bigcup_{q\in F} V_q$$

and p and F are below r, then

$$V_r \subset V_p \setminus \bigcup_{q\in F} V_q$$

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Let $(r_n)_{n\in\omega}$ be an enumeration of the α-th level. Then we can use the $\mathcal{C}_{2k}$'s to cover it. By the previous observation, it will cover every element above the α-th level as well.
But below the α-th level, there are only countably many elements. Therefore we can use the $\mathcal{C}_{2k+1}$'s to cover them.

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It can seem that we could use the same idea to prove that TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$.
Everything seems to work, except how to compute the α.
And as we will see, in fact TWO has no winning strategy (and since ONE also does not have one, the game is undetermined).

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We did this for the 0-th inning. We can do the same thing for the other (possible) innings and define:

$$K_s = \bigcap_{\mathcal{C} \in \mathcal{O}} \sigma(s \frown \mathcal{C})$$

where s is a finite sequence of open coverings. As before, $|K_s| \leq 1$.

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What we will do in the following is to reduce how many s's we have to keep, while the union of the respective K_s's still covers the whole X.

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So each $|K_s| \leq 1$ and here are countably many K_s's that are enough to cover the space. Thus the space has to be countable.

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(Note that this solves the case for the Suslin tree.)

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(Note that this solves the case for the Suslin tree.)
But let us prove it by a different kind of argument. We will prove the following:

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(which is basically the same thing because of the duality)

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- And so on.

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You can prove this by definition and using regularity (it is a nice proof). We will do something different.

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Then if we repeat the argument of the "all possible answers are countable" and "for each answer there is a question", we obtain that X is the union of countably many K_s's.

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In this case, we have the following:

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Actually, the same is true if we change $\mathsf{G}_2(\mathcal{O}, \mathcal{O})$ to $\mathsf{G}_{\mathrm{k}}(\mathcal{O}, \mathcal{O})$ for any $k \in \mathbb{N}_{>0}$.