Day 3 – Tightness games

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The tightness game

Definition (Moore–Mrówka 1964)

A topological space X has **countable tightness** at a point $p \in X$ if, for every $A \subseteq X$ with $p \in \overline{A}$, there is a countable $B \subseteq A$ such that $p \in \overline{B}$.

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Definition (Sakai 1988)

A topological space X has **countable strong fan tightness** at a point $p \in X$ if, for every sequence $(A_n)_{n\in\omega}$ of subsets of X satisfying $p \in \overline{A_n}$, we can select $x_n \in A_n$ for $n \in \omega$ so that $p \in \overline{\{x_n : n \in \omega\}}$.

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Note that

X has countable strong fan tightness at $p \Leftrightarrow \mathsf{S}_1(\Omega_p, \Omega_p)$

where $\Omega_p := \{A \subseteq X : p \in \overline{A}\}$.

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When $\text{TWO} \uparrow \mathsf{G}_1(\Omega_p, \Omega_p)$

Didn't I see that before?

Theorem (Gruenhage 1976)

If X is a separable regular space and $p \in X$ is such that $\text{TWO} \uparrow \mathsf{G}_1(\Omega_p, \Omega_p)$, then X is first-countable at p.

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Proof.

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- The winner is $\textsc{One}$ if $\{x_n : n \in \omega\} \in \Omega_p$, and $\textsc{Two}$ otherwise.

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The previous lemma is the key to proving that if TWO has a winning strategy in $\mathsf{G}_1(\Omega_p, \Omega_p)$, then ONE has a winning strategy for the neighbourhood-point game.

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The previous lemma is the key to proving that if TWO has a winning strategy in $\mathsf{G}_1(\Omega_p, \Omega_p)$, then ONE has a winning strategy for the neighbourhood-point game. (Exercise!)

The other implications

On the other hand, if σ is a strategy for Two in the neighbourhood-point game and $\langle V_0, \ldots, V_n \rangle \in {\tau_p}^{<\omega}$, then

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With this, we can prove that, if $\textsc{Two}$ has a winning strategy in the neighbourhood-point game, then $\textsc{One}$ has a winning strategy for $\mathsf{G}_1(\Omega_p, \Omega_p)$.

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With this, we can prove that, if Two has a winning strategy in the neighbourhood-point game, then One has a winning strategy for $\mathsf{G}_1(\Omega_p, \Omega_p)$. The other two implications are quite straightforward. Therefore, we obtain:

Proposition

The games $\mathsf{G}_1(\Omega_p, \Omega_p)$ and neighbourhood-point are dual.

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Exercise

Fill in the details of the proof.

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By induction, there are only countably many possibilities for One to play in the whole game.

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Note that, if $\textsc{One}$ plays B, then $\textsc{Two}$ can pick an $x \in (B \smallsetminus \overline{V}) \cap D$. And playing like this, $\{x_n : n \in \omega\} \notin \Omega_p$.

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- $\psi(\langle x_0, \ldots, x_n \rangle) =$
  $\bigcap\{\varphi(\langle x_{i_0}, x_{i_1}, \ldots, x_{i_m} \rangle) \;:\; m \leq n \;\&\; (i_k)_{k \leq m}$ is an increasing sequence in $\{0, \ldots, n\}\}$.

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Now let $U \in \tau_p$ be such that $I = \{n \in \omega : x_n \notin U\}$ is infinite, and let $I = \{i_k : k \in \omega\}$ be an increasing enumeration.

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is a play of the neighbourhood-open game

A variation of these games

Theorem (Gruenhage 1976)

ONE has a winning strategy in the neighbourhood-point game
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is a play of the neighbourhood-open game (why?) in which ONE makes use of the winning strategy φ; hence, $p \in \overline{\{x_{i_k} : k \in \omega\}}$. Contradiction! □

A variation of these games

Are these games in fact equivalent?
What about player TWO*?*

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Example (Gruenhage 2006)

There is a countable space with only one non-isolated point on which TWO has a winning strategy in the neighbourhood-point convergence game but not on the neighbourhood-point game.

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We will skip the details of this example to talk about another example of a countable space with only one non-isolated point...

How does $G_1(\Omega_p, \Omega_p)$ *relate to* $S_1(\Omega_p, \Omega_p)$*?*

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Example ((adapted from) Scheepers 1997)

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$$V(F, n) := Y \setminus \bigcup_{f \in F} \{f \restriction_m : m \in \omega \;\&\; m \geq n\}$$

for $F \subset \omega^\omega$ finite and $n \in \omega$.

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Proposition

$\textsc{One} \uparrow \mathsf{G}_1(\Omega_p, \Omega_p)$ on Y.

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Proof.

ONE initiates the game by playing $A_0 = \{\langle k \rangle : k \in \omega\} \in \Omega_p$.

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ONE initiates the game by playing $A_0 = \{\langle k \rangle : k \in \omega\} \in \Omega_p$. If TWO picks $\langle k_0 \rangle \in A_0$, ONE's move in the next inning is $A_1 = \{\langle k_0, k \rangle : k \in \omega\} \in \Omega_p$.

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$\textsc{One} \uparrow \mathsf{G}_1(\Omega_p, \Omega_p)$ on Y.

Proof.

$\textsc{One}$ initiates the game by playing $A_0 = \{\langle k \rangle : k \in \omega\} \in \Omega_p$. If $\textsc{Two}$ picks $\langle k_0 \rangle \in A_0$, $\textsc{One}$'s move in the next inning is $A_1 = \{\langle k_0, k \rangle : k \in \omega\} \in \Omega_p$. If $\textsc{Two}$ picks $\langle k_1 \rangle \in A_1$, $\textsc{One}$'s move in the next inning is $A_2 = \{\langle k_0, k_1, k \rangle : k \in \omega\} \in \Omega_p$. By proceeding in this fashion, we obtain $f = \langle k_0, k_1, k_2, \dots \rangle \in \omega^\omega$ such that $V(\{f\}, 0) = Y \setminus \{f \restriction_m : m \in \omega\}$ is an open neighbourhood of p that does not contain any of the points played by $\textsc{Two}$. Thus, $\textsc{One}$ wins the play. □

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Proof.

We only have to observe that Two can play in such a way that, at the end of each inning $n \in \omega$, the set of all the points she has chosen includes a set $\{s_0, \ldots, s_n\}$ such that every branch of $\omega^{<\omega}$ contains at most one element of this set. (Note that this implies that Two wins the play.)

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We can do so by induction.

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We can do so by induction. The case $n = 0$ is trivial. Now suppose this is true for some $n \in \omega$.
If there is some point $a \in A_{n+1}$ that lies in a branch of $\omega^{<\omega}$ that misses the set $\{s_0, \ldots, s_n\}$, let Two pick this point together with any other one.

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For every space X and $p \in X$,

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How does $\mathsf{G}_1(\Omega_p, \Omega_p)$ relate to $\mathsf{S}_1(\Omega_p, \Omega_p)$?

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$\textsc{One} \uparrow \mathsf{G}_1(\Omega_p, \Omega_p)$ on Y.

Proposition

$\textsc{Two} \uparrow \mathsf{G}_2(\Omega_p, \Omega_p)$ on Y.

Theorem (García-Ferreira–Tamariz-Mascarúa 1995)

For every space X and $p \in X$,

$$\mathsf{S}_2(\Omega_p, \Omega_p) \iff \mathsf{S}_1(\Omega_p, \Omega_p).$$

Corollary

$\mathsf{S}_1(\Omega_p, \Omega_p)$ and $\textsc{One} \not\uparrow \mathsf{G}_1(\Omega_p, \Omega_p)$ are not equivalent.

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