

Day 3 – Tightness games

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The tightness game

Definition (Moore–Mrówka 1964)

A topological space X has **countable tightness** at a point $p \in X$ if, for every $A \subseteq X$ with $p \in \overline{A}$, there is a countable $B \subseteq A$ such that $p \in \overline{B}$.

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Definition (Sakai 1988)

A topological space X has **countable strong fan tightness** at a point $p \in X$ if, for every sequence $(A_n)_{n \in \omega}$ of subsets of X satisfying $p \in \overline{A_n}$, we can select $x_n \in A_n$ for $n \in \omega$ so that $p \in \overline{\{x_n : n \in \omega\}}$.

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Note that

X has countable strong fan tightness at $p \Leftrightarrow S_1(\Omega_p, \Omega_p)$

where $\Omega_p := \{A \subseteq X : p \in \overline{A}\}$.

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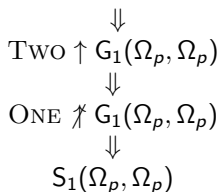
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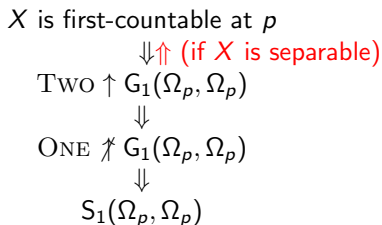
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When $\text{TWO} \uparrow G_1(\Omega_p, \Omega_p)$

Didn't I see that before?

Theorem (Gruenhage 1976)

If X is a separable regular space and $p \in X$ is such that $\text{TWO} \uparrow G_1(\Omega_p, \Omega_p)$, then X is first-countable at p .

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$$\tau_p := \{U \in \tau : p \in U\}$$

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Lemma

If $\sigma : \Omega_p^{<\omega} \rightarrow X$ is a strategy for TWO in $G_1(\Omega_p, \Omega_p)$, then for every $\langle A_0, \dots, A_n \rangle \in \Omega_p^{<\omega}$ there is an open neighbourhood of p such that, for every $x \in U$, there is an $A \in \Omega_p$ such that $\sigma(\langle A_0, \dots, A_n, A \rangle) = x$.

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(Exercise!)

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On the other hand, if σ is a strategy for TWO in the neighbourhood-point game and $\langle V_0, \dots, V_n \rangle \in \tau_p^{<\omega}$, then

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The games $G_1(\Omega_p, \Omega_p)$ and neighbourhood-point are dual.

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Fill in the details of the proof.

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By induction, there are only countably many possibilities for ONE to play in the whole game.

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Note that, if ONE plays B , then TWO can pick an $x \in (B \setminus \overline{V}) \cap D$. And playing like this, $\{x_n : n \in \omega\} \notin \Omega_p$.

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- $\psi(\langle x_0, \dots, x_n \rangle) =$
 $\bigcap \{ \varphi(\langle x_{i_0}, x_{i_1}, \dots, x_{i_m} \rangle) : m \leq n \quad \&$
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Then

$$\langle \varphi(\langle \rangle), x_{i_0}, \varphi(\langle x_{i_0} \rangle), x_{i_1}, \varphi(\langle x_{i_0}, x_{i_1} \rangle), x_{i_2}, \dots, \varphi(\langle x_{i_0}, x_{i_1}, \dots, x_{i_k} \rangle), x_{i_{k+1}}, \dots \rangle$$

is a play of the neighbourhood-open game

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Suppose not. Let $\langle x_0, x_1, \dots, x_n, \dots \rangle$ be TWO's moves in a play of this game in which TWO has beaten the strategy ψ .

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Then

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A variation of these games

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Example (Gruenhage 2006)

There is a countable space with only one non-isolated point on which TWO has a winning strategy in the neighbourhood-point convergence game but not on the neighbourhood-point game.

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We will skip the details of this example to talk about another example of a countable space with only one non-isolated point...

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- basic neighbourhoods of p are of the form

$$V(F, n) := Y \setminus \bigcup_{f \in F} \{f \upharpoonright_m : m \in \omega \text{ \& } m \geq n\}$$

for $F \subset \omega^\omega$ finite and $n \in \omega$.

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We only have to observe that TWO can play in such a way that, at the end of each inning $n \in \omega$, the set of all the points she has chosen includes a set $\{s_0, \dots, s_n\}$ such that every branch of $\omega^{<\omega}$ contains at most one element of this set. (Note that this implies that TWO wins the play.)

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Corollary

$S_1(\Omega_p, \Omega_p)$ and ONE $\not\uparrow$ $G_1(\Omega_p, \Omega_p)$ are **not** equivalent.

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