Remarks on the Menger property of $C_p(X, 2)$

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Contents

Introduction

All spaces are Tychonoff.

$C_p(X)$: the space of all real valued continuous functions with the topology of pointwise convergence.

$C_p(X, \mathbb{I}) = \{f \in C_p(X) : f(X) \subset \mathbb{I}\}$ $\quad \mathbb{I} = [0, 1]$

$C_p(X, 2) = \{f \in C_p(X) : f(X) \subset 2\}$ $\quad 2 = \{0, 1\}$

Definition 1.1

X has **the Menger property** (or, X is Menger) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exist finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{V}_n$ is a cover of X.

$C_p(X)$ is σ-compact if and only if X is finite (N.V. Velichko).

Theorem 1.2 (A.V. Arhangel'skii, 1986)

$C_p(X)$ is Menger if and only if X is finite.

$C_p(X)$ is σ-compact if and only if X is finite (N.V. Velichko).

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$C_p(X, \mathbb{I})$ is Menger if and only if X is discrete.

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Problem 1.4

For a zero-dimensional space X, when is $C_p(X, 2)$ Menger?

Fact 1.5

If K is a zero-dimensional compact metrizable space, then $C_p(K, 2)$ is countable.

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If $A(\omega_1) = D(\omega_1) \cup \{\infty\}$ is the one-point compactification of the discrete space $D(\omega_1)$ of cardinality ω_1. Then $C_p(A(\omega_1), 2)$ is σ-compact.

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Theorem 1.7 (in Arhangel'skii's book, p.158)

For a zero-dimensional compact space K, $C_p(K, 2)$ is σ-compact if and only if K is Eberlein compact.

A space is said to be **simple** if it has exactly one non-isolated point.

Fact 1.8

Let $X = D \cup \{p\}$ be a simple space. If $\chi(p, X) = \omega$, then $C_p(X, 2)$ is σ-compact.

Proof.

Let $\{U_n : n \in \omega\}$ be a countable base at p.
$K_n = \{f \in 2^X : f(x) = 0$ for any $x \in U_n\}$
K_n is compact and
$\bigcup_{n\in\omega} K_n = \{f \in C_p(X, 2) : f(p) = 0\}$. □

The results of Bernal-Santos and Tamariz-Mascarúa

A. Contreras-Carreto, A. Tamariz-Mascarúa, On some generalizations of compactness in spaces $C_p(X, 2)$ and $C_p(X, \mathbb{Z})$, Bol. Soc. Mat. Mexicana 9 (2003), 291–308.

D. Bernal-Santos, Á. Tamariz-Mascarúa, The Menger property on $C_p(X, 2)$, Top. Appl. 183 (2015), 110–126.

Definitions: For a space X and a subset $A \subset X$,

X' is the space of all non-isolated points of X.

A is **bounded in** X if for any continuous $f : X \to \mathbb{R}$, $f(A)$ is bounded in $\mathbb{R}$.

A family $\mathcal{A}$ of subsets of a space X is a π-**network at** $x \in X$ if every neighborhood of x contains some member in $\mathcal{A}$.

X has **countable fan-tightness for finite sets** if for any $x \in X$ and any π-network $\mathcal{A}_n$ at x consisting of non-empty finite subsets of X, there exist finite $\mathcal{B}_n \subset \mathcal{A}_n$ such that $\bigcup\{\mathcal{B}_n : n \in \omega\}$ is a π-network at x.

All spaces are zero-dimensional.

Theorem 2.1 (Bernal-Santos, Tamariz-Mascarúa, 2015)

If $C_p(X,2)$ is Menger, then every finite power of X has countable fan-tightness for finite sets.

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$C_p(S_\omega, 2)$ is a separable metric space which is not Menger.

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Theorem 2.3 (B.T., 2015)

If $C_p(X, 2)$ is Menger, then X' is bounded in X.

Theorem 2.4 (B.T., 2015)

Let X be a subspace of $C_p(Y)$, where every finite power of Y is Menger. If X' is compact, then $C_p(X,2)^n$ is Menger for each $n \in \mathbb{N}$.

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Problem 2.5 (B.T., Problem 8.9, 2015)

If $C_p(X,2)$ is Menger, then is $C_p(X,2)^n$ Menger for $n \geq 2$.

Special cases:

all spaces are zero-dimensional.

Metrizable spaces:

Lemma 2.6 (Contreras-Carreto, Tamariz-Mascarúa, 2003)

If X is a subspace of an Eberlein compact space and X' is compact, then $C_p(X, 2)$ is σ-compact.

Corollary 2.7

If X is metrizable and X' is compact, then $C_p(X, 2)$ is σ-compact.

Theorem 2.8 (B.T., 2015)

The following are equivalent for a metrizable space X.

(1) $C_p(X, 2)$ *is Menger,*

(2) $C_p(X, 2)^n$ *is Menger for each* $n \in \mathbb{N}$,

(3) $C_p(X, 2)$ *is* σ*-compact,*

(4) X' *is compact.*

Proof.

(4)$\rightarrow$ (3) is due to Corollary 2.7.
(3)$\rightarrow$(2)$\rightarrow$(1) are obvious.
(1)$\rightarrow$(4) is due to Theorem 2.3. □

Simple countable spaces:

Let p be a free filter on ω.

Let $X(p) = \omega \cup \{p\}$ be a countable simple space

Each point of ω is isolated, and
a neighborhood at p is $F \cup \{p\}$, where $F \in p$.

Theorem 2.9 (B.T., 2015)

The following are equivalent for $X(p)$.

(1) $C_p(X(p), 2)$ *is Menger,*

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Proof.

(1)→(3) is due to Theorem 2.1. (3)→(2) $\cdots$. □

Remark 2.10

$C_p(X(p), 2)$ *is Menger if and only if* $p \subset 2^\omega$ *is Menger.*

Corollary 2.11 (B.T., 2015)

For $p \in \omega^*$, the following are equivalent.

(1) $C_p(X(p), 2)$ *is Menger,*

(2) $C_p(X(p), 2)^n$ *is Menger for each* $n \in \mathbb{N}$*,*

(3) p *is a strong P-point.*

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Remark 2.12

· *A strong P-point is a P-point, and not rapid (C. Laflamme, 1989).*

· $PR(X(p))$ *is selectively separable if and only if* p *is a strong P-point.*

Discussions

Let X be zero-dimensional and let $A \subset X$ be closed.

Assume A has an outer base of clopen sets in X.

Let $X/A = (X \setminus A) \cup \{A\}$ be the quotient space obtained by collapsing A to singleton, then $\underline{X/A \text{ is zero-dimensional and Tychonoff}}$.

Let $2 = (\mathbb{Z}_2, +)$, then $\underline{C_p(X, 2) \text{ is a topological group}}$.

$C_p(X, 2; A, 0) =$
$\{f \in C_p(X, 2) : f(a) = 0 \text{ for any } a \in A\}$:
a closed subgroup of $C_p(X, 2)$.

$C_p(X/A, 2; \{A\}, 0) =$
$\{g \in C_p(X/A, 2) : g(A) = 0\}$:
a closed subgroup of $C_p(X/A, 2)$.

Lemma 3.1

$C_p(X, 2; A, 0) \simeq C_p(X/A, 2; \{A\}, 0)$ *as a topological group.*

Definition 3.2

A subset $A \subset X$ is a **retract** if there exists a continuous map $r : X \to A$ such that $r \restriction A = id_A$. The map $r : X \to A$ is called a **retraction**.

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Proposition 3.3

If $A \subset X$ is retract, then we have the following isomorphisms as a topological group.

$$\begin{aligned} C_p(X,2) &\simeq C_p(A,2) \times C_p(X,2;A,0) \\ &\simeq C_p(A,2) \times C_p(X/A,2;\{A\},0) \end{aligned}$$

Let $r : X \to A$ be a retraction.

We define a map
$\varphi : C_p(X, 2) \to C_p(A, 2) \times C_p(X, 2; A, 0)$ as follows:

$$\varphi(f) = (f{\restriction}A, (f{\restriction}A \circ r) + f).$$

φ is continuous:

$f \mapsto (f{\restriction}A, f) \mapsto (f{\restriction}A \circ r, f) \mapsto (f{\restriction}A \circ r) + f$

$$\varphi : C_p(X,2) \to C_p(A,2) \times C_p(X,2;A,0) :$$
$$\varphi(f) = (f{\restriction}A, (f{\restriction}A \circ r) + f)$$

φ is one-to-one:

Let $f, g \in C_p(X,2)$ and assume $f \neq g$.
If $f(a) \neq g(a)$ for some $a \in A$, then $\varphi(f) \neq \varphi(g)$.

Assume $f{\restriction}A = g{\restriction}A$. Take a point $b \in X \setminus A$ with $f(b) \neq g(b)$.

Let $f(b) = 0,\ g(b) = 1$.
$$((f{\restriction}A \circ r) + f)(b) = f(r(b)) + f(b) = f(r(b))$$
$$((g{\restriction}A \circ r) + g)(b) = g(r(b)) + g(b) = f(r(b)) + 1$$

$\varphi : C_p(X,2) \to C_p(A,2) \times C_p(X,2;A,0)$:
$\varphi(f) = (f{\restriction}A, (f{\restriction}A \circ r) + f)$

φ is onto:

Take any $(p,q) \in C_p(A,2) \times C_p(X,2;A,0)$.

Let $h = (p \circ r) + q \in C_p(X,2)$.

For any $a \in A$,
$h(a) = ((p \circ r) + q)(a) = (p \circ r)(a) = p(a)$, thus
$h{\restriction}A = p$.

Moreover,
$((h{\restriction}A) \circ r) + h = (p \circ r) + (p \circ r) + q = q$.

$\varphi : C_p(X,2) \to C_p(A,2) \times C_p(X,2;A,0)$:

$\varphi(f) = (f{\restriction}A, (f{\restriction}A \circ r) + f)$

φ^{-1} is continuous:

$(p,q) \mapsto (p \circ r, q) \mapsto p \circ r + q$. □

Fact 3.4

(well-known) Every compact metrizable subset A of a zero-dimensional space X is a retract of X.

Note that X' has a outer base of clopen sets in X.

Proposition 3.5 (The first idea)

If X' is compact metrizable, then we have the following isomorphism as a topological group.

$$C_p(X,2) \simeq C_p(X',2) \times C_p(X/X',2;\{X'\},0)$$

Definition 3.6

A space X is **projectively Menger** if every second countable continuous image of X is Menger.

Proposition 3.7 (Lj. Koćinac, 2006, The second idea)

A space is Menger if and only if it is Lindelöf and projectively Menger.

Fact 3.8

Let $X = D \cup \{p\}$ *be simple and let* $p \in A \subset X$. *Then* $\pi_A : C_p(X,2) \to C_p(A,2)$ *is onto.*

Proof.

Let $f \in C_p(A,2)$ and let $p \in f^{-1}(i)$, where $i = 0$, or 1. Then $f^{-1}(1-i)$ is clopen in X.

Define $g \in C_p(X,2)$ as follows:
$g(x) = 1 - i$ for $x \in f^{-1}(1-i)$,
$g(x) = i$ for $x \in X \setminus f^{-1}(1-i)$.

Then $\pi_A(g) = f$. □

Lemma 3.9

Let $X = D \cup \{p\}$ *be simple. TFAE.*

(1) $C_p(X, 2)$ *is projectively Menger.*

(2) *for each* $C \in [D]^\omega$ *with* $p \in \overline{C}$, $C \cup \{p\}$ *has countable fan-tightness for finite sets.*

Proof.

(1)→(2): Consider $\pi_{C \cup \{p\}} : C_p(X, 2) \to C_p(C \cup \{p\}, 2)$ *and use Fact 3.8 and Theorem 2.9.*

(2)→(1): Take any continuous $\varphi : C_p(X, 2) \to Y$. *There exists a countable* $p \in A \subset X$ *and a continuous* $\psi : \pi_A(C_p(X, 2)) \to Y$ *such that* $\varphi = \psi \circ \pi_A$. *Use Fact 3.8 and Theorem 2.9.* □

Recall that if X' is compact metrizable,

$$C_p(X,2) \simeq C_p(X',2) \times C_p(X/X',2;\{X'\},0)$$

Proposition 3.10

If X' is compact metrizable, TFAE.

(1) $C_p(X,2)$ *is projectively Menger.*

(2) *for each* $C \in [X \setminus X']^\omega$ *with* $\overline{C} \cap X' \neq \emptyset$, $C \cup \{X'\}$ *has countable fan-tightness for finite sets (in* X/X'*).*

$$C_p(X,2) \simeq C_p(X',2) \times C_p(X/X',2;\{X'\},0)$$

Proposition 3.11

If X' is compact metrizable, TFAE.

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(3) $C_p(X,2)$ *is Lindelöf and* X/X' *has countable fan-tightness for finite sets.*

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(3) $C_p(X, 2)$ *is Lindelöf and* X/X' *has countable fan-tightness for finite sets.*

(4) $C_p(X/X', 2; \{X'\}, 0)$ *is Lindelöf and* X/X' *has countable fan-tightness for finite sets.*

A space is said to be **Dieudonné complete** if it can be embedded as a closed subset into a Cartesian product of metrizable spaces.

Every paracompact space, or realcompact space is Dieudonné complete.

Every closed bounded subset of a Dieudonné complete space is compact (*e.x.*, see Engelking's book, p.464, 8.5.13.(b) (3)).

Proposition 3.12

Let X be a Dieudonné complete space with a G_δ-diagonal. TFAE

(1) $C_p(X, 2)$ *is Menger.*

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Corollary 3.13

For a countable space X*, the following are equivalent.*

(1) $C_p(X, 2)$ *is Menger,*

(2) X' *is compact metrizable and* X/X' *has countable fan-tightness for finite sets.*

Corollary 3.14

For a countable space X, if $C_p(X,2)$ is Menger, so is every finite power of $C_p(X,2)$.

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Remark 3.15

Under $\mathfrak{b} = \mathfrak{d}$, there exist countable simple spaces $X_0 = \omega \cup \{p\},\ X_1 = \omega \cup \{q\}$ such that both $C_p(X_0,2)$ and $C_p(X_1,2)$ are Menger, but $C_p(X_0,2) \times C_p(X_1,2)$ is not Menger.