Mathematics and the Borel Conjecture

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Frontiers of Selection Principles
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F. Rothberger: 1902 - 2000

"A turning point in Dr. Rothberger's career came in 1937, when he visited Warsaw, which at that time was the focal point for research in set theory. Here he came into contact with Wladislaw Sierpiński and his associates $\cdots$. This experience led him to continue and extend his own research in this area, and the outcome was a series of research papers of major importance $\cdots$. This work was recognized in 1977 when a symposium was held in his honour at the University of Toronto."

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Outline

The Original Definition

Definition (1919, Borel)

A set X of real numbers is *strong measure zero* if:

For each sequence $(\epsilon_n : n < \omega)$ of positive real numbers, there exists a sequence $(x_n : n < \omega)$ of elements of X such that:

$$X \subseteq \bigcup_{n<\omega} (x_n - \epsilon_n, x_n + \epsilon_n).$$

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Interpretations

1 $(\mathbb{R}, |\cdot|)$ is a metric space, and we are using neighborhoods of points.

2 $(\mathbb{R}, +)$ is a topological group, and we are translating neighborhoods of the identity element.

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A selection principle

The symbol

$$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$$

means:
For each sequence $(A_n : n < \omega)$ of elements of $\mathcal{A}$,
there is a sequence $(b_n : n < \omega)$ such that:
(a) For each n, $b_n \in A_n$ and
(b) $\{b_n : n < \omega\}$ is an element of $\mathcal{B}$.

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Some terminology

A subset X of a space Y is *relatively Rothberger* if $\mathsf{S}_1(\mathcal{O}, \mathcal{O}_X)$ holds.

A space Y is *Rothberger* if $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ holds.

Fix a topological group $(G, *)$ with topology τ. Fix a τ-neighborhood U of the identity e_G.

$\mathcal{O}(U) := \{x * U : x \in G\}$ is an open cover of G.

$\mathcal{O}_{nbd,\tau} = \{\mathcal{O}(U) :\ U \text{ a } \tau\text{-neighborhood of } e_G\}$.

With τ clear from context, write $\mathcal{O}_{nbd}$ for $\mathcal{O}_{nbd,\tau}$.

A subset X of a topological group $(G, \odot)$ is *Rothberger bounded* if $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}_X)$ holds.

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The infinite game $\mathsf{G}_1^{\alpha}(\mathcal{O}, \mathcal{O})$

$\mathcal{O}$ denotes the collection of all open covers of a space X.
$\alpha > 0$ is an ordinal.

Players ONE and TWO play an inning per $\gamma < \alpha$.

In inning γ ONE first chooses an element $O_\gamma \in \mathcal{O}$.

TWO responds by choosing a $T_\gamma \in O_\gamma$.

A play

$$O_0,\ T_0,\ \cdots,\ O_\gamma,\ T_\gamma,\ \cdots \quad \gamma < \alpha$$

is won by TWO if $\{T_\gamma : \gamma < \alpha\}$ is an open cover of X.
Else, ONE wins.

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Characterizations: Metric Spaces

Theorem

For a subset X of a σ-compact metric space Y the following are equivalent:

1. *X is strong measure zero*
2. *$S_1(\mathcal{O}, \mathcal{O}_X)$ (X is relatively Rothberger in Y)*
3. *ONE has no winning strategy in the game $G_1^\omega(\mathcal{O}, \mathcal{O}_X)$*
4. *For each positive integer k, $\Omega \rightarrow (\mathcal{O}_X)^2_k$*

Characterizations: Metrizable topological groups

Theorem (Fremlin-Kysiak)

If $(G, \odot)$ is a locally compact metrizable topological group and X is a subset of G, the following are equivalent:

1. *X is strong measure zero*
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Characterizations: T_0 Topological Groups

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1. $S_1(\mathcal{O}_{\text{nbd}}, \mathcal{O}_X)$ *(X is Rothberger bounded in G)*
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Possibilities - 1

Theorem (Comfort)

Let $(G_i : i \in I)$ be a family of countable topological groups. Endow the product $\prod_{i \in I} G_i$ with the G_δ topology. Then the subgroup

$$G := \{f \in \prod_{i \in I} G_i : |\{j \in I : f(j) = id_j\}| < \aleph_0\}$$

is a Lindelöf P-group.

There is for each uncountable cardinal κ a Rothberger (bounded) group of cardinality κ

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Equivalences of Borel's Conjecture

Theorem

The following statements are equivalent:

1. *Every strong measure zero set of real numbers is countable*
2. *Every strong measure zero metric space is countable*
3. *Every Rothberger set of real numbers is countable*
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Equivalences of Borel's Conjecture

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Theorem (Sierpiński (1928), Gödel (1938))

CON(ZFC) $\Rightarrow$ CON(ZFC + $\neg$BC)

Theorem (Laver (1976))

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Outline

Version A: Tall and Usuba

Definition

A Lindelöf space is indestructibly Lindelöf if ONE has no winning strategy in the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.

(Tall and Usuba, 2012)

$\mathsf{BC}_{\aleph_1}^{TU}$ *is the statement:*
A Lindelöf space is indestructible if, and only if, each continuous image of it in $[0, 1]^{\aleph_1}$ *is of cardinality at most* $\aleph_1$.

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Consistency of $\mathrm{BC}^{TU}_{\aleph_1}$.

Theorem (Dias, Tall and Usuba)

The following theories are equiconsistent:

1. ZFC + $\mathrm{BC}^{TU}_{\aleph_1}$.
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$\mathrm{BC}^{TU}_{\aleph_1} \Rightarrow \mathrm{CH}$

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Version B: Halko.

Let κ be an uncountable cardinal.

For each f in ${}^{\kappa}2$ and each $\alpha < \kappa$ define:

$$U(f,\alpha) = \{g \in {}^{\kappa}2 : g\lceil_{\alpha} = f\lceil_{\alpha}\}.$$

Let τ_κ be the topology generated by $\{U(f,\alpha) : f \in {}^{\kappa}2,\ \alpha < \kappa\}$.

NOTES:

1) τ_κ refines the usual Tychonoff product topology τ.

2) $({}^{\kappa}2, \oplus)$ is a topological group in τ_κ, as well as τ.

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A notion of κ-strong measure zero for $\kappa > \aleph_0$ regular.

$X \subset {}^{\kappa}2$ is *κ-strong measure zero* if:

For each increasing sequence $(\alpha_\nu : \nu < \kappa)$ in κ there exists a sequence $(f_\nu : \nu < \kappa)$ in X such that

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(Halko, 1996)

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BC_κ^H and cardinal arithmetic.

Theorem (Halko-Shelah)

If $\kappa = \kappa^{<\kappa} = \mu^+$, then BC_κ^H is false

Thus: $2^\kappa = \kappa^+$ implies $\neg\mathrm{BC}_{\kappa^+}^H$.

Corollary

$$\mathrm{BC}_{\aleph_1}^{TU} \Rightarrow \neg\mathrm{BC}_{\aleph_1}^H.$$

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BC_κ^H questions

Problem (Halko-Shelah)

Is κ strongly inaccessible $+\mathrm{BC}_\kappa^H$ consistent?

Version C: Galvin - Borel Conjecture for Rothberger bounded sets

For topological group $(G, *)$ and cardinal number $\lambda > \aleph_0$ define:

Definition

BC$(G, < \lambda)$: Each Rothberger bounded subset of G is of cardinality less than λ.

Halko calls BC$(({}^{\omega_1}2, \oplus, \tau_{\omega_1}), < \aleph_2)$ the

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(Guran) A group is said to be $\aleph_0$-bounded if each element of $\mathcal{O}_{\text{nbd}}$ has a countable subset that covers the group.

Theorem (Guran)

A topological group is $\aleph_0$-bounded if, and only if, it embeds as topological group into a product of second countable topological groups

For each infinite cardinal κ the group $({}^{\kappa}2, \oplus)$ is $\aleph_0$-bounded in the product topology.

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Let κ be an infinite cardinal number.

BC_κ : *Each Rothberger bounded subset of an $\aleph_0$-bounded group of weight κ, has cardinality at most κ.*

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Does any instance of BC_κ hold?

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A crucial lemma

Lemma

Let κ be an infinite cardinal number. Let $(G_\alpha : \alpha < \kappa)$ be topological groups. Let a subset X of $G = \prod_{\alpha<\kappa} G_\alpha$ be given. The following are equivalent:

1. *X is Rothberger bounded.*
2. *For each countable set $C \subseteq \kappa$ the set $X\lceil_C$ is a Rothberger bounded subset of $G\lceil_C$.*

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2. *For each countable set $C \subseteq \kappa$ the set $X\lceil_C$ is a Rothberger bounded subset of $G\lceil_C$.*

Basic facts about failure of BC_κ

Theorem (G-S)

BC_κ *cannot first fail at* κ *when:*

1. κ *is a singular strong limit cardinal of uncountable cofinality.*
2. κ *is an ineffable cardinal.*

Theorem (G-S)

Let $\kappa > \lambda$ *be infinite cardinal numbers.*
If $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$ *holds, then* $\mathrm{BC}_\lambda \Rightarrow \mathrm{BC}_\kappa$.

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A fundamental combinatorial statement

Let $\kappa \geq \lambda$ be cardinal numbers.

$\mathcal{F} \subset \mathcal{P}(\kappa)$ is a (κ, λ) Kurepa family if:

1. $|\mathcal{F}| > \kappa$ and
2. For each $A \subset \kappa$ with $|A| < \lambda$,

$$|\{X \cap A : X \in \mathcal{F}\}| \leq |A|.$$

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Theorem (G-S)

Assume BC. *Then for each uncountable cardinal κ the following are equivalent:*

1. BC_{κ}.
2. $\mathsf{BC}({}^{\kappa}2, \kappa)$.
3. *Each Rothberger bounded subgroup of $({}^{\kappa}2, \oplus)$ has cardinality at most κ.*
4. $\neg\mathsf{KH}(\kappa, \aleph_1)$.

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The case $\kappa = \aleph_1$

Theorem (Halko, thesis)

$\mathrm{BC}(({}^{\omega_1}2, \oplus, \tau_{\omega_1}), < \aleph_2)$ *if, and only if, there is no* ω_1*-Kurepa tree.*

Corollary (G-S)

Assume BC. *The following statements are equivalent:*

1. $\mathrm{BC}_{\aleph_1}$.
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$2^{\aleph_1} = \aleph_2 \Rightarrow \neg\mathrm{BC}^H_{\aleph_1}$.

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Further Consistency Results

Theorem (G-S)

If it is consistent that there is a 1-inaccessible cardinal with countably many inaccessible cardinals above it, then it is consistent that BC_κ first fails at $\aleph_\omega$.

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If it is consistent that there is a 2-huge cardinal, then it is consistent that $\mathrm{BC}_{\aleph_\omega}$ holds.

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Lower bounds for the case of $\aleph_\omega$

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If $2^{\aleph_0} < \aleph_\omega$ and $\mathrm{BC}({}^{\aleph_\omega}2, \aleph_\omega)$, then Projective Determinacy holds.

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How far can we go?

Problem

Is $\mathrm{ZFC} + (\forall\kappa)(\mathrm{BC}_\kappa)$ *consistent relative to the consistency of some large cardinal?*

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If it is consistent that there is a 3-huge cardinal, then it is consistent that BC_κ *holds for a proper class of cardinals* κ *such that* $\mathrm{cof}(\kappa) = \omega$.

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