Mathematics and the Borel Conjecture - II

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Frontiers of Selection Principles
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Outline

Version B: Halko.

Let κ be an uncountable cardinal.

For each f in ${}^{\kappa}2$ and each $\alpha < \kappa$ define:

$$U(f,\alpha) = \{g \in {}^{\kappa}2 : g\lceil_{\alpha} = f\lceil_{\alpha}\}.$$

Let τ_{κ} be the topology generated by $\{U(f,\alpha) : f \in {}^{\kappa}2, \alpha < \kappa\}$.

NOTES:

1) τ_{κ} refines the usual Tychonoff product topology τ.

2) $({}^{\kappa}2, \oplus)$ is a topological group in τ_{κ}, as well as τ.

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A notion of κ-strong measure zero for $\kappa > \aleph_0$ regular.

$X \subset {}^{\kappa}2$ is *κ-strong measure zero* if:

For each increasing sequence $(\alpha_\nu : \nu < \kappa)$ in κ there exists a sequence $(f_\nu : \nu < \kappa)$ in X such that

$$X \subseteq \bigcup_{\nu < \kappa} U(f_\nu, \alpha_\nu).$$

(BC_κ^H)

Each κ - strong measure zero subset of ${}^{\kappa}2$ has cardinality $\leq \kappa$.

Theorem (Halko)

$2^{\aleph_1} = \aleph_2 \Rightarrow \neg BC_{\aleph_1}^H$.

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Borel Conjecture for Rothberger bounded sets

X is Rothberger bounded in the group $(G,*)$ if $S_1(\mathcal{O}_{nbd},\mathcal{O})$ holds.

For topological group $(G,*)$ and cardinal number $\lambda > \aleph_0$ define:

Definition

BC$(G,<\lambda)$: Each Rothberger bounded subset of G is of cardinality less than λ.

Halko calls BC$(({}^{\omega_1}2,\oplus,\tau_{\omega_1}),<\aleph_2)$ the *weak generalized Borel conjecture*.

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Let κ be an infinite cardinal number.

BC_κ : *Each Rothberger bounded subset of an $\aleph_0$-bounded group of weight κ, has cardinality at most κ.*

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The connection with fundamental combinatorics

Theorem (G-S)

Assume BC. *Then for each uncountable cardinal κ the following are equivalent:*

1. BC_κ.
2. $BC({}^{\kappa}2, \kappa)$.
3. *Each Rothberger bounded subgroup of $({}^{\kappa}2, \oplus)$ has cardinality at most κ.*
4. $\neg KH(\kappa, \aleph_1)$.

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The case $\kappa = \aleph_1$

Theorem (Halko, thesis)

$BC(({}^{\omega_1}2, \oplus, \tau_{\omega_1}), < \aleph_2)$ *if, and only if, there is no* ω_1*-Kurepa tree.*

Corollary

Assume BC*. The following statements are equivalent:*

1. $BC_{\aleph_1}$.
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Assume the consistency of the existence of a 1-inaccessible cardinal.
Then it is consistent that $\mathsf{BC} + \mathsf{BC}_{\aleph_1} + 2^{\aleph_1} = \aleph_2$ *holds.*

Theorem (Halko)

$2^{\aleph_1} = \aleph_2 \Rightarrow \neg \mathsf{BC}^H_{\aleph_1}$.

Corollary

If it is consistent that there is a 1-inaccessible cardinal then it is consistent that $\mathsf{BC} + w - \mathsf{BC}^H_{\aleph_1} + \neg \mathsf{BC}^H_{\aleph_1}$.

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Outline

The Original Definition

Definition (1919, Borel)

A set X of real numbers is *strong measure zero* if:

For each sequence $(\epsilon_n : n < \omega)$ of positive real numbers, there exists a sequence $(x_n : n < \omega)$ of elements of X such that:

$$X \subseteq \bigcup_{n<\omega} (x_n - \epsilon_n, x_n + \epsilon_n).$$

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Borel's Conjecture

Conjecture (Borel, 1919)

If a set X of real numbers is strong measure zero then X is countable.

BC denotes Borel's Conjecture.

Equivalences of Borel's Conjecture

Theorem

The following are equivalent:

1. *Every strong measure zero set of real numbers is countable*
2. *Each Rothberger bounded subset of a group of countable weight is countable.*
3. *For every Rothberger bounded subset of a group of countable weight TWO has a winning strategy in the game $G_1^\omega(\Omega,\Gamma)$.*

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Example 1

Theorem (Comfort)

Let $(G_i : i \in I)$ be a family of countable topological groups. Endow the product $\prod_{i\in I} G_i$ with the G_δ topology. Then the subgroup

$$G := \{f \in \prod_{i\in I} G_i : |\{j \in I : f(j) \neq id_j\}| < \aleph_0\}$$

is a Lindelöf P-group.

There is for each uncountable cardinal κ a Rothberger (bounded) group of cardinality κ

Theorem (S)

TWO has a winning strategy in the game $G_1(\Omega,\Gamma)$ on G.

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Let $(G_i : i \in I)$ be a family of σ-compact topological groups. The subgroup

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Questions

For infinite cardinal numbers κ, does BC_κ imply that:

1. *For each Rothberger bounded subset X of an $\aleph_0$-bounded group of weight κ the game $G_1^\omega(\mathcal{O}_{nbd},\mathcal{O}_X)$ is determined?*
2. *Each Rothberger bounded subset X of an $\aleph_0$-bounded group of weight κ has property $S_1(\Omega,\Omega)$?*
3. *For each Rothberger bounded subset X of an $\aleph_0$-bounded group of weight κ ONE has no winning strategy in the game $G_1^\omega(\Omega,\Omega)$?*
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The Ellentuck topology based on a countable set.

For a countable set $A = (a_n : n \in \mathbb{N})$, enumerated bijectively and $s,\ T \subset A$:

$s < T$ if $a_n \in s$ and $a_m \in T \Rightarrow n < m$.

$$[s, T] = \{C \subset A : s \subset C \subset s \cup T\}.$$

$$B|s = \{a_n \in B : s < \{a_n\}\}.$$

$\{[s,\ T] : s < T$ and $T \subset A\}$ is a basis for the Ellentuck topology on $[A]^{\aleph_0}$.

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$E(\mathcal{A}, \mathcal{B})$:

For each countably infinite $A \in \mathcal{A}$ and for each set $R \subset [A]^{\aleph_0} \cap \mathcal{B}$ (1)$\Rightarrow$(2) holds, where:

1. R has the Baire property in the Ellentuck topology on $[A]^{\aleph_0} \cap \mathcal{B}$.
2. For each $S \subset A$ with $S \in \mathcal{A}$ and each finite subset s of A, there is an infinite $B \subset S|s$ with $B \in \mathcal{B}$ such that $[s, B] \cap \mathcal{B} \subseteq R$ or $[s, B] \cap \mathcal{B} \cap R = \emptyset$.

Thus, $E([\mathbb{N}]^{\aleph_0}, [\mathbb{N}]^{\aleph_0})$ is Ellentuck's theorem.

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For positive integers n and k and for each countable $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \cdots, k\}$ there is a $B \in [A]^{\aleph_0} \cap \mathcal{B}$ and an $i \in \{1, \cdots, k\}$ such that f has value i on $[B]^n$.

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Mathias Reals Posets

For a (bijectively enumerated) countable ω-cover A of space X define:

$$\mathcal{M}_A := \{(s, C) : s \in [A]^{<\aleph_0} \text{ and } C \subset A|s \text{ and } C \in \Omega_X\}$$

For $(s_1,\ C_1)$ and $(s_2,\ C_2)$ elements of $\mathcal{M}_A$, define:

$$(s_1, C_1) \prec (s_2, C_2) \text{ if: } s_2 \subset s_1 \text{ and } C_1 \subset C_2 \text{ and } s_1 \setminus s_2 \subset C_2|s_2$$

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Some equivalences

Theorem

Let X be a space for which $S_1(\Omega,\Omega)$ implies ONE has no winning strategy in $G_1^\omega(\Omega,\Omega)$. The following are equivalent:

1. $S_1(\Omega,\Omega)$
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3. $(\forall n)(\forall k)\Omega \to (\Omega)^n_k$.
4. $E(\Omega,\Omega)$.
5. *For each countable $A \in \Omega$: for each sentence Ψ in the $\mathcal{M}_A$-forcing language, and for each $(s, B) \in \mathcal{M}_A$, there is a $C \subset B$ with $C \in \Omega_X$ such that $(s, C) \Vdash \Psi$, or $(s, C) \Vdash \neg\Psi$.*

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