Compactness properties defined by open-point games

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Two well-known compactness proprties

Let us recall two well-known compactness-type properties.

Definition

A topological space X is called:

(i) *sequentially compact* if every sequence in X has a convergent subsequence;

(ii) *pseudocompact* if every continuous real-valued function defined on X is bounded;

Making (i) into a selection principle:
A space X is sequentially compact if for every sequence $\{A_n : n \in \mathbb{N}\}$ of singletons, one can choose a point $x_n \in A_n$ in such a way that the resulting sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.

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A space X is *selectively sequentially pseudocompact* provided that, for every sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X, one can choose a point $x_n \in U_n$ in such a way that the resulting sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.

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Main features of selective sequential pseudocompactness

X is sequentially compact

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- Sequential compactness is countably productive but not productive in general.
- Pseudocompactness is not even finitely productive (although it becomes productive in topological groups).
- Selective sequential pseudocompactness is productive.
- Dyadic spaces (in particular, compact groups) are selectively sequentially pseudocompact.

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X has a dense sequentially compact subspace

$\downarrow (b)$

X has a dense subspace that is relatively sequentially compact

$\downarrow (c)$

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$\downarrow (d)$

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- the Tychonoff plank shows thatarrow (a) is not reversible.
- the Mrówka space shows that arrow (b) is not reversible.
- a pseudocompact space X such that all countable subsets of X are closed shows that arrow (d) is not reversible.

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- We are going to show that arrow (c) is not reversible either.
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Open-point games $OP(X, \mathcal{S})$, for a fixed topological property $\mathcal{S}$ of sequences

Definition

- Fix a topological property $\mathcal{S}$ of sequences.
- Let X be a topological space.
- An *open-point topological game $OP(X, \mathcal{S})$ on X* between Player A and Player B is played as follows.
- In an nth move, Player A chooses a non-empty open subset U_n of X. In response, Player B selects a point $x_n \in U_n$.
- There is an nth move for every $n \in \mathbb{N}$, and then the game is over.
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- When $\mathcal{S}$ is the property "the sequence has a subsequence converging in X", we shall call the game $OP(X, \mathcal{S})$ the *selectively sequentially pseudocompact game on X* and denote this game by $Ssp(X)$ for breivity. Player B wins the play

$$U_0, x_0, U_1, x_1, \ldots, U_n, x_n, \ldots$$

in the game $Ssp(X)$ if the sequence $\{x_n : n \in \mathbb{N}\}$ has a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ that converges to some point $x \in X$.

- When $\mathcal{S}$ is the property "the sequence has an accumulation point in X", we shall call the game $OP(X, \mathcal{S})$ the *selectively pseudocompact game on X* and denote this game by $Sp(X)$ for breivity. Player B wins the play

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in the game $Sp(X)$ if there exists $x \in X$ such that the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite for every neighbourhood U of x.

Two concrete properties $\mathcal{S}$: Games $Ssp(X)$ and $Sp(X)$

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The game $OP(X, \mathcal{S})$ leads to four new topological properties

Player B has a stationary winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 1

Player B has a winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 2

Player A does not have a winning strategy in $OP(X, \mathcal{S})$

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Player A does not have a stationary winning strategy in $OP(X, \mathcal{S})$

Selective $\mathcal{S}$-pseudocompactness

Definition

A space X is *selectively $\mathcal{S}$-pseudocompact* if for every sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can select a point $x_n \in U_n$ for $n \in \mathbb{N}$ in such a way that the resulting sequence $\{x_n : n \in \mathbb{N}\}$ satisfies property $\mathcal{S}$ in X.

Proposition

Let $\mathcal{S}$ be a topological property of sequences which is implied by sequential compactness. If Player A does not have a stationary winning strategy in $OP(X, \mathcal{S})$, then X is selectively $\mathcal{S}$-pseudocompact.

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Proposition

Let $\mathcal{S}$ be a topological property of sequences which is implied by sequential compactness. If X is not selectively $\mathcal{S}$-pseudocompact, then Player A has a stationary winning strategy in $OP(X, \mathcal{S})$.

Suppose that X is not selectively $\mathcal{S}$-pseudocompact. Since selective sequential pseudocompactness implies property $\mathcal{S}$ by our assumption, X is not selectively sequentially pseudocompact. Then there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of *pairwise disjoint* non-empty open subsets of X such that for every choice of a point $x_n \in U_n$ for $n \in \mathbb{N}$, the sequence $\{x_n : n \in \mathbb{N}\}$ does not satisfy $\mathcal{S}$.

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So we have a sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X such that for every choice of a point $x_n \in U_n$ for $n \in \mathbb{N}$, the sequence $\{x_n : n \in \mathbb{N}\}$ does not satisfy $\mathcal{S}$.

Define the strategy σ for the Player A as follows.

- At his first move, Player A chooses U_1.
- Let $(x_1, x_2, \ldots, x_k)$ be the first k moves of the Player B. If $x_k \in \bigcup_{n \in \mathbb{N}} U_n$, then Player A chooses U_{m+1}, where m is the unique integer such that $x_k \in U_m$. Othwerise, Player A playes U_1.

Since the strategy σ depends only on the last move of the opponent, σ is stationary.

Clearly, σ is the winning strategy for A, because irrespective of the moves $(x_1, x_2, \ldots, x_k)$ by Player B, Player A ends up selecting the sequence $\{U_n : n \in \mathbb{N}\}$. Since $x_n \in U_n$ for every n by the rules of the game, this implies that $\{x_n : n \in \mathbb{N}\}$ does not satisfy property $\mathcal{S}$. This means that Player A wins the game.

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- Let $(x_1, x_2, \ldots, x_k)$ be the first k moves of the Player B. If $x_k \in \bigcup_{n \in \mathbb{N}} U_n$, then Player A chooses U_{m+1}, where m is the unique integer such that $x_k \in U_m$. Othwerise, Player A playes U_1.

Since the strategy σ depends only on the last move of the opponent, σ is stationary.

Clearly, σ is the winning strategy for A, because irrespective of the moves $(x_1, x_2, \ldots, x_k)$ by Player B, Player A ends up selecting the sequence $\{U_n : n \in \mathbb{N}\}$. Since $x_n \in U_n$ for every n by the rules of the game, this implies that $\{x_n : n \in \mathbb{N}\}$ does not satisfy property $\mathcal{S}$. This means that Player A wins the game.

So we have a sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X such that for every choice of a point $x_n \in U_n$ for $n \in \mathbb{N}$, the sequence $\{x_n : n \in \mathbb{N}\}$ does not satisfy $\mathcal{S}$.

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- At his first move, Player A chooses U_1.
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A topological characterization of Player B having a stationary winning strategy in $OP(X, \mathcal{S})$

Definition

A subspace Y of X *relatively satisfies property* $\mathcal{S}$ *in* X if every sequence of points of Y satisfies property $\mathcal{S}$ in X.

Proposition

Player B has a stationary winning strategy in $OP(X, \mathcal{S})$ if and only if X has a dense subspace which relatively satisfies $\mathcal{S}$ in X.

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A diagram

Definition

A space X *satisfies* $\mathcal{S}$ if every sequence in X satisfies $\mathcal{S}$.

$$\begin{array}{c} X \text{ satisfies } \mathcal{S} \\ \Big\downarrow (a) \\ X \text{ has a dense subspace satisfying } \mathcal{S} \\ \Big\downarrow (b) \\ X \text{ has a dense subspace that relatively satisfies } \mathcal{S} \text{ in } X \\ \Big\updownarrow \\ \text{Player } B \text{ has a stationary winning strategy in } OP(X, \mathcal{S}) \end{array}$$

The diagram continued: the fine structure of arrow (c)

Player *B* has a stationary winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 1

Player *B* has a winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 2

Player *A* does not have a winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 3

Player *A* does not have a stationary winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 4

X is selectively $\mathcal{S}$-pseudocompact

Arrow (1) is not reversible: The hardest example

Theorem

There exists a locally compact, first-countable, zero-dimensional space X such that Player B has a winning strategy but has no stationary winning strategy in $Ssp(X)$. In fact, Player B does not have a stationary winning startegy even in the "weaker" game $Sp(X)$.

Corollary

Let $\mathcal{S}$ be any topological property of sequences between countable compactness and sequential compactness. Let X be a locally compact, first-countable zero-dimensional space we have constructed above. Then Player B has a winning strategy in $OP(X, \mathcal{S})$ but has no stationary winning strategy in $OP(X, \mathcal{S})$.

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Arrow (4) is not reversible: The easier example

Example (A.J. Berner)

- For each $C \in [\omega_1]^\omega$ define

$$Y(C) = \{f \in 2^{\omega_1} : f(\alpha) = 0 \text{ if } \alpha \notin C \cup \{\sup C + 1\}$$

$$\text{and } f(\sup C + 1) = 1\}.$$

- A. J. Berner proved that Y is a pseudocompact space that does not contain a dense relatively countably compact subspace.

Theorem

Let $\mathcal{S}$ is a topological property of sequences between countable compactness and sequential compactness. Then the above Y is selectively $\mathcal{S}$-pseudocompact and Player A has a stationary winning strategy in $OP(Y, \mathcal{S})$.

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X is selectively $\mathcal{S}$-pseudocompact

Let $\mathcal{S}$ is a topological property of sequences between countable compactness and sequential compactness. Then arrows 1 and 4 are not reversible. We do not know if arrows 2 and 3 are reversible.

Question

Let $\mathcal{S}$ is a topological property of sequences between countable compactness and sequential compactness. Is the game $OP(X, \mathcal{S})$ determined?

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$$\downarrow 1$$

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$$\downarrow 2$$

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X is selectively $\mathcal{S}$-pseudocompact

Question

Let $\mathcal{S}$ is a topological property of sequences between countable compactness and sequential compactness. Are any of these arrows reversible for compact spaces X?

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$\downarrow$ 1

Player B has a winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 2

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Question

Let $\mathcal{S}$ is a topological property of sequences between countable compactness and sequential compactness. Are any of these arrows reversible for topological groups X?

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$\downarrow$ 1

Player B has a winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 2

Player A does not have a winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 3

Player A does not have a stationary winning strategy in $OP(X, \mathcal{S})$

$\downarrow$ 4

X is selectively $\mathcal{S}$-pseudocompact

Question

Let $\mathcal{S}$ is a topological property of sequences between countable compactness and sequential compactness. Are any of these arrows reversible for the G-valued function spaces $X = C_p(Y, G)$, where G is a topological group?

Construction of the hardest example

Theorem

There exists a locally compact, first-countable, zero-dimensional space X such that Player B has a winning strategy in Ssp(X) but does not have a stationary winning startegy even in the "weaker" game Sp(X).

Recall that:

- in the Ssp(X) game, Player B has a winning strategy if the sequence $\{x_n : n \in \mathbb{N}\}$ he picks has a subsequence converging to some point of X;
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The van Douwen MAD family of functions from $\mathbb{N}$ to $\mathbb{N}$

- A family $\mathscr{F} \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be *almost disjoint* if the set $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite whenever $f, g \in \mathscr{F}$ are distinct.
- Following D. Raghavan, we shall say that p is an *infinite partial function* if $p \in \mathbb{N}^P$ for some infinite subset P of $\mathbb{N}$.
- An almost disjoint family $\mathscr{F} \subseteq \mathbb{N}^{\mathbb{N}}$ is a *Van Douwen MAD family* if for every infinite partial function $p \in \mathbb{N}^P$, there is $f \in \mathscr{F}$ such that the set $\{n \in P : p(n) = f(n)\}$ is infinite.

We fix a van Douwen MAD family $\mathscr{F}$ of size $\mathfrak{c}$; the existence of such a family (in ZFC!) was proved by D. Raghavan (2007).

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Taking essentially injective subfamily $\mathscr{G}$ of $\mathscr{F}$

For every $g \in \mathscr{F}$, let

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : A \text{ is infinite and } g\restriction_A \text{ is an injection}\}. \qquad (1)$$

Define

$$\mathscr{G} = \{g \in \mathscr{F} : \mathcal{I}_g \neq \emptyset\}.$$

For every $g \in \mathscr{G}$, selecting a MAD subfamily $\mathcal{A}_g$ of $\mathcal{I}_g$

For every $g \in \mathscr{F}$, let

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : A \text{ is infinite and } g \restriction_A \text{ is an injection}\}. \qquad (2)$$

Define

$$\mathscr{G} = \{g \in \mathscr{F} : \mathcal{I}_g \neq \emptyset\}.$$

For every $g \in \mathscr{G}$, use Zorn's lemma to fix a maximal almost disjoint subfamily $\mathcal{A}_g$ of $\mathcal{I}_g$; that is,

(a) $A \cap A'$ is finite whenever $A, A' \in \mathcal{A}_g$ are distinct;

(b) if $T \in \mathcal{I}_g$, then $T \cap A$ is infinite for some $A \in \mathcal{A}_g$.

Clearly, $\mathcal{A}_g \neq \emptyset$.

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Clealry, $\mathcal{A}_g \neq \emptyset$.

Fixing a MAD family $\mathscr{H}$ of injections from $\mathbb{N}$ to $\mathfrak{c}^+$

Let D be a set of cardinality $\mathfrak{c}^+$. Applying Zorn's lemma, we can fix a family $\mathscr{H} \subseteq D^{\mathbb{N}}$ having the following properties:

(i) each $h \in \mathscr{H}$ is injective;

(ii) if $h_1, h_2 \in \mathscr{H}$ and $h_1 \neq h_2$, then the set $h_1(\mathbb{N}) \cap h_2(\mathbb{N})$ is finite;

(iii) if S is an infinite subset of D, then $S \cap h(\mathbb{N})$ is infinite for some $h \in \mathscr{H}$.

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Definition of the set R

Let C be the Cantor set. Define

$$R = \{(c, g, h, A) : c \in C, g \in \mathscr{G}, h \in \mathscr{H}, A \in \mathcal{A}_g\} \tag{3}$$

Lemma

If $(c, g, h, A) \in R$, then $g \restriction_A$ is an injection.

This immediately follows from $A \in \mathcal{A}_g \subseteq \mathcal{I}_g$ and (2).

Selecting clopen rings V_n^c in the Cantor set C converging to $c \in C$

For every $c \in C$, fix a strictly decreasing base $\{W_n^c : n \in \mathbb{N}\}$ at c consisting of clopen subsets of C such that $W_0^c = X$, and let $V_n^c = W_n^c \setminus W_{n+1}^c$ for every $n \in \mathbb{N}$.

Lemma

For every $c \in C$, the family $\{V_n^c : n \in \mathbb{N}\}$ is a partition of $C \setminus \{c\}$ consisting of non-empty clopen subsets of C.

Recall that C is the Cantor set.

Lemma

For every $c \in C$, the family $\{V_n^c : n \in \mathbb{N}\}$ is a partition of $C \setminus \{c\}$ consisting of non-empty clopen subsets of C.

Consider the discrete topology on D, and let $M = C \times D$ be equipped with the Tychonoff product topology. For

$$(c, g, h, A) \in R = \{(c, g, h, A) : c \in C, g \in \mathscr{G}, h \in \mathscr{H}, A \in \mathcal{A}_g\}$$

and $n \in \mathbb{N}$, both

$$M^n_{c,g,h} = V^c_n \times \{h \circ g(n)\} \tag{4}$$

and

$$O^n_{c,g,h,A} = \bigcup_{m \in A, m > n} M^m_{c,g,h} \tag{5}$$

are clopen subsets of M.

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and

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are clopen subsets of M.

Lemma

Suppose that $(c, g_i, h, A_i) \in R$ *for* $i = 1, 2$ *and* $O^n_{c,g_1,h,A_1} \cap O^n_{c,g_2,h,A_2} \neq \emptyset$. *Then there exists* $m \in A_1 \cap A_2$ *such that* $m > n$ *and* $h \circ g_1(m) = h \circ g_2(m)$.

By (5), there exists $m_i \in A_i$ satisfying $m_i > n$ for $i = 1, 2$ such that $M^{m_1}_{c,g_1,h,A_1} \cap M^{m_2}_{c,g_2,h,A_2} \neq \emptyset$. Since $M^{m_i}_{c,g_i,h,A_i} = V^c_{m_i} \times \{h \circ g_i(m_i)\}$ for $i = 1, 2$ by (4), we deduce that $V^c_{m_1} \cap V^c_{m_2} \neq \emptyset$ and $h \circ g_1(m_1) = h \circ g_2(m_2)$. From the former inequality and the partition lemma we conclude that $m_1 = m_2 = m$, so from the latter equality we get $h \circ g_1(m) = h \circ g_2(m)$.

Lemma

If $(c_1, g_1, h_1, A_1), (c_2, g_2, h_2, A_2) \in R$ *are distinct, then*

$$O^n_{c_1,g_1,h_1,A_1} \cap O^n_{c_2,g_2,h_2,A_2} = \emptyset \tag{6}$$

for some $n \in \mathbb{N}$*.*

We consider four cases.

Case 1. $c_1 \neq c_2$. Since $\{W_n^{c_i} : n \in \mathbb{N}\}$ is a strictly decreasing base at c_i for $i = 1, 2$, there exists $n \in \mathbb{N}$ such that $W_n^{c_1} \cap W_n^{c_2} = \emptyset$. Moreover, $\bigcup_{m>n} V_m^{c_i} \subseteq W_n^{c_i}$ for $i = 1, 2$. Combining this with (4) and (5), we get (6).

Case 2. $h_1 \neq h_2$. From item (ii) of the definition of $\mathscr{H}$, we conclude that the set $h_1(\mathbb{N}) \cap h_2(\mathbb{N})$ is finite. Since h_1 is injective by item (i) of the definition of $\mathscr{H}$, the subset $h_1^{-1}(h_1(\mathbb{N}) \cap h_2(\mathbb{N}))$ of $\mathbb{N}$ is finite, so $h_1^{-1}(h_1(\mathbb{N}) \cap h_2(\mathbb{N})) \subseteq n_0$ for some $n_0 \in \mathbb{N}$. Now $\{h_1(k) : k > n_0\} \cap h_2(\mathbb{N}) = \emptyset$. Since $g_1 \restriction_{A_1}$ is an injection, we can take $n \in \mathbb{N}$ such that $g_1(m) > n_0$ for every $m \in A_1$ with $m > n$. Hence,

$$\{h_1 \circ g_1(m) : m \in A_1, m > n\} \cap \{h_2 \circ g_2(m) : m \in A_2, m > n\} \subseteq$$

$$\{h_1(k) : k > n_0\} \cap h_2(\mathbb{N}) = \emptyset.$$

Combining this with (4) and (5), we get (6).

Case 3. $c_1 = c_2 = c$, $h_1 = h_2 = h$ and $g_1 \neq g_2$. Since $g_1, g_2 \in \mathscr{G} \subseteq \mathscr{F}$ are distinct and the family $\mathscr{F}$ is almost disjoint, the set $\{k \in \mathbb{N} : g_1(k) = g_2(k)\}$ is finite, so we can fix $n \in \mathbb{N}$ such that $g_1(m) \neq g_2(m)$ whenever $m \in \mathbb{N}$ and $m > n$. Since h is injective by item (i) of the definition of $\mathscr{H}$, we have

$$h \circ g_1(m) \neq h \circ g_2(m) \text{ for every } m > n. \tag{7}$$

Suppose that (6) fails. Then the assumption of the previous lemma is satisfied. Let m be as in the conclusion of this lemma. Then $m > n$ and $h \circ g_1(m) = h \circ g_2(m)$, in contradiction with (7). This contradiction shows that (6) holds.

Case 4. $c_1 = c_2 = c, h_1 = h_2 = h, g_1 = g_2 = g$ *and* $A_1 \neq A_2$. Since $A_i \in \mathcal{A}_{g_i} = \mathcal{A}_g$ for $i = 1, 2$, the set $A_1 \cap A_2$ is finite by item (a) of the definition of $\mathcal{A}_g$. Hence, $A_1 \cap A_2 \subseteq n$ for some $n \in \mathbb{N}$. Suppose that (6) fails. Then the assumption of the previous lemma is satisfied for $g_1 = g_2 = g$. Let m be as in the conclusion of this lemma. Then $m > n$ and $m \in A_1 \cap A_2$, in contradiction with $A_1 \cap A_2 \subseteq n$. This contradiction shows that (6) holds.

$$R = \{(c, g, h, A) : c \in C, g \in \mathscr{G}, h \in \mathscr{H}, A \in \mathcal{A}_g\}$$

Without loss of generality, we shall assume that $M \cap R = \emptyset$. Consider the topology on the set

$$X = M \cup R$$

defined by declaring $M = C \times D$ to be an open subspace of X and taking the family

$$\mathcal{B}_{c,g,h,A} = \{B^n_{c,g,h,A} : n \in \mathbb{N}\}, \tag{8}$$

where

$$B^n_{c,g,h,A} = \{(c, g, h, A)\} \cup O^n_{c,g,h,A} \quad \text{for } n \in \mathbb{N}, \tag{9}$$

as a local base at each point $(c, g, h, A) \in R$.

Clealry, M is dense in X and R is a closed discrete subspace of X. The fact that X is first countable is straightforward from the definition.

Lemma

X is Hausdorff.

First, since M is Hausdorff and open in X, any two points in M can be separated by disjoint open subsets of X.

Next, let $(c_1, g_1, h_1, A_1), (c_2, g_2, h_2, A_2) \in R$ be distinct. Let $n \in \mathbb{N}$ be as in the conclusion of the previous lemma. Since $M \cap R = \emptyset$, it follows from (6) and (9) that $B^n_{c_1,g_1,h_1,A_1} \cap B^n_{c_2,g_2,h_2,A_2} = \emptyset$. Since $B^n_{c_i,g_i,h_i,A_i} \in \mathcal{B}_{c_i,g_i,h_i,A_i}$ for $i = 1, 2$ by (8), the points (c_1, g_1, h_1, A_1) and (c_2, g_2, h_2, A_2) of X can be separated by two disjoint basic open subsets of X.

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Finally, assume that $(c_0, d) \in M$ and $(c, g, h, A) \in R$. Since h is injective by item (i) of the definition of $\mathscr{H}$, there is at most one m_0 such that $d = h(m_0)$. (If there exists no such m_0, we define $m_0 = 1$.) Since $g \restriction_A$ is injective, there is at most one $n \in A$ such that $g(n) = m_0$. (If no such n exists, we define $n = 1$.) If $m \in A$ and $m > n$, then $h \circ g(m) \neq d$ by our choice of n, so $(C \times \{d\}) \cap M^m_{c,g,h} = \emptyset$ by (4). Combining this with (5), we get $(C \times \{d\}) \cap O^n_{c,g,h,A} = \emptyset$, so $(C \times \{d\}) \cap B^n_{c,g,h,A} = \emptyset$ by (9). It follows from (8) and (9) that $B^n_{c,g,h,A}$ is an open neighbourhood of (c, g, h, A). Since $C \times \{d\}$ is open in M and M is open in X, the former set is open in X. We conclude that $C \times \{d\}$ and $B^n_{c,g,h,A}$ are disjoint open subsets of X that separate (c_0, d) and (c, g, h, A).

This finishes the proof that X is Hausdorff.

Finally, assume that $(c_0, d) \in M$ and $(c, g, h, A) \in R$. Since h is injective by item (i) of the definition of $\mathscr{H}$, there is at most one m_0 such that $d = h(m_0)$. (If there exists no such m_0, we define $m_0 = 1$.) Since $g \restriction_A$ is injective, there is at most one $n \in A$ such that $g(n) = m_0$. (If no such n exists, we define $n = 1$.) If $m \in A$ and $m > n$, then $h \circ g(m) \neq d$ by our choice of n, so $(C \times \{d\}) \cap M^m_{c,g,h} = \emptyset$ by (4). Combining this with (5), we get $(C \times \{d\}) \cap O^n_{c,g,h,A} = \emptyset$, so $(C \times \{d\}) \cap B^n_{c,g,h,A} = \emptyset$ by (9). It follows from (8) and (9) that $B^n_{c,g,h,A}$ is an open neighbourhood of (c, g, h, A). Since $C \times \{d\}$ is open in M and M is open in X, the former set is open in X. We conclude that $C \times \{d\}$ and $B^n_{c,g,h,A}$ are disjoint open subsets of X that separate (c_0, d) and (c, g, h, A).

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This finishes the proof that X is Hausdorff.

Lemma

X is locally compact and zero-dimensional.

Let $(c, d) \in M$ be arbitrary. Consider an open subset U of X containing (c, d). Then $U \cap M$ is an open subset of M containing (c, d). Since M is locally compact and zero-dimensional, there exists a clopen compact subset K of M such that $(c, d) \in K \subseteq U \cap M$. Since K is open in M and M is open in X, K is open in X. Since K is compact and X is Hausdorff, K is closed in X. Thus, K is a compact clopen subset of X such that $(c, d) \in K \subseteq U$.

Let $(c, g, h, A) \in R$ be arbitrary. Since $\mathcal{B}_{c,g,h,A}$ is a local base of X at (c, g, h, A), in view of (8), it suffices to check that each $B^k_{c,g,h,A}$ is a compact clopen subset of X.
Fix $k \in \mathbb{N}$. By definition, $B^k_{c,g,h,A}$ is an open subset of X. Let us check that $B^k_{c,g,h,A}$is a closed subset of X. Since $B^k_{c,g,h,A} \cap M = O^k_{c,g,h,A}$ by (9), the latter set is closed in M and M is open in X, it follows that every point $(c, d) \in M \setminus B^k_{c,g,h,A}$ has an open neighbourhood in X disjoint from $B^k_{c,g,h,A}$. Let $(c', g', h', A') \in R \setminus B^k_{c,g,h,A}$. Then $(c, g, h, A) \neq (c', g', h', A')$, so we can apply Claim 12 to find $n \in \mathbb{N}$ such that $O^n_{c,g,h,A} \cap O^n_{c',g',h',A'} = \emptyset$.

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If $n \leq k$, then $O^k_{c,g,h,A} \subseteq O^n_{c,g,h,A}$ and $O^k_{c',g',h',A'} \subseteq O^n_{c',g',h',A'}$ by (5), so the sets $O^k_{c,g,h,A}$ and $O^k_{c',g',h',A'}$ are disjoint, which implies $B^k_{c,g,h,A} \cap B^k_{c',g',h',A'} = \emptyset$ by (9).
Assume now that $k < n$. Then

$$B^k_{c,g,h,A} \setminus B^n_{c,g,h,A} = O^k_{c,g,h,A} \setminus O^n_{c,g,h,A} = \bigcup_{m \in A, k < m \leq n} M^m_{c,g,h} = \bigcup_{m \in A, k < m \leq n} V^c_m \times \{h \circ g(m)\}$$

by (4), (5) and (9), so this set is compact, and thus closed in X.
Now $B^n_{c',g',h',A'} \setminus (B^k_{c,g,h,A} \setminus B^n_{c,g,h,A})$ is an open neighbourhood of (c', g', h', A') disjoint from $B^k_{c,g,h,A}$.
We have proved that $B^k_{c,g,h,A}$ is a clopen subset of X. Since $B^k_{c,g,h,A} \setminus B^n_{c,g,h,A}$ is compact whenever $k < n$, it follows that each $B^k_{c,g,h,A}$ is compact.

Lemma

For every $c^* \in C$, the set $Z_{c^*} = \{c^*\} \times D$ is discrete and closed in X.

Clearly, Z_{c^*} is discrete in M, and thus also in X. Furthermore, Z_{c^*} is obviously closed in M. So it remains only to show that no point $(c, g, h, A) \in X \setminus M = R$ lies in the closure of Z_{c^*}.

Fix a point $(c, g, h, A) \in R$. Since $\{V_n^c : n \in \mathbb{N}\}$ is a partition of $C \setminus \{c^*\}$, there exists at most one $n \in \mathbb{N}$ such that $c^* \in V_n^c$. (If no such n exists, we define $n = 1$.) By (4), (5) and (9), $B^n_{c,g,h,A}$ does not intersect Z_{c^*}. Since $B^n_{c,g,h,A} \in \mathcal{B}_{c,g,h,A}$ by (8), it is an open neighbourhood of (c, g, h, A) in X.

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Lemma

Player B does not have a stationary winning strategy in $Sp(X)$.

Recall that Player B has a stationary winning strategy in $Sp(X)$ precisely when X has a dense relatively countably compact subset. Therefore, it suffices to show that X does not have a dense relatively countably compact subset. Let Y be a dense subset of X. For every $\beta < \mathfrak{c}^+$, the set $U_\beta = C \times \{\beta\}$ is open in X. Hence, there is an element $c_\beta \in C$ such that $(c_\beta, \beta) \in U_\beta \cap Y$ for every $\beta < \mathfrak{c}^+$. Since $|C| = \mathfrak{c}$, there exist $c^* \in C$ and a faithfully indexed set $\{\beta_n : n \in \mathbb{N}\}$ such that $c^* = c_{\beta_n}$ for every $n \in \mathbb{N}$. Then $S = \{(c^*, \beta_n) : n \in \mathbb{N}\} \subseteq Z_{c^*}$. Since Z_{c^*} is a closed discrete subspace of X by the previous lemma, S has no accumulation points in X. Since S is contained in Y, we conclude that Y is not relatively countably compact in X.

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Recall that Player B has a stationary winning strategy in $Sp(X)$ precisely when X has a dense relatively countably compact subset. Therefore, it suffices to show that X does not have a dense relatively countably compact subset. Let Y be a dense subset of X. For every $\beta < \mathfrak{c}^+$, the set $U_\beta = C \times \{\beta\}$ is open in X. Hence, there is an element $c_\beta \in C$ such that $(c_\beta, \beta) \in U_\beta \cap Y$ for every $\beta < \mathfrak{c}^+$. Since $|C| = \mathfrak{c}$, there exist $c^* \in C$ and a faithfully indexed set $\{\beta_n : n \in \mathbb{N}\}$ such that $c^* = c_{\beta_n}$ for every $n \in \mathbb{N}$. Then $S = \{(c^*, \beta_n) : n \in \mathbb{N}\} \subseteq Z_{c^*}$. Since Z_{c^*} is a closed discrete subspace of X by the previous lemma, S has no accumulation points in X. Since S is contained in Y, we conclude that Y is not relatively countably compact in X.

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Lemma

Suppose that J is an infinite subset of $\mathbb{N}$ and $\{c_j : j \in J\}$ is a sequence in C converging to $c \in C$ such that $c_l \neq c_m$ whenever $l \neq m$. Then there exist strictly increasing functions $j : \mathbb{N} \to J$ and $k : \mathbb{N} \to \mathbb{N}$ such that $c_{j(m)} \in V^c_{k(m)}$ for every $m \in \mathbb{N}$.

Without loss of generality, we shall assume that $c_j \neq c$ for every $j \in \mathbb{N}$. Since $\{V_n^c : n \in \mathbb{N}\}$ is a partition of $C \setminus \{c\}$, each c_j is contained in exactly one element $V^c_{n_j}$ of this partition. Moreover, since V_n^c is a clopen subset of C and the sequence $\{c_j : j \in J\}$ converges to $c \notin V_n^c$, each V_n^c contains at most finitely many elements of the sequence $\{c_j : j \in J\}$.

By induction on $m \in \mathbb{N}$, we shall define $j(m) \in J$ and $k(m) \in \mathbb{N}$ such that:

(1_m) $c_{j(m)} \in V^c_{k(m)}$;

(2_m) if $m \geq 2$, then $j(m) > j(m-1)$ and $k(m) > k(m-1)$.

Let $j(1) \in J$ be arbitrary. Define $k(1) = n_{j(1)}$. Then (1_1) and (2_1) hold.

Let $m \geq 2$ and suppose that $j(s) \in J$ and $k(s) \in \mathbb{N}$ satisfying (1_s) and (2_s) have already been defined for every $s \leq m-1$. The set $\bigcup_{n \leq k(m-1)} V^c_n$ contains only finitely many elements of the sequence $\{c_j : j \in J\}$. Since J is infinite, we can find $j(m) \in J$ such that $j(m-1) < j(m)$ and $c_{j(m)} \notin \bigcup_{n \leq k(m-1)} V^c_n$. Let $k(m) = n_{j(m)}$. Then $c_{j(m)} \in V^c_{n_{j(m)}} = V^c_{k(m)}$, which implies $k(m-1) < k(m)$. Thus, (1_m) and (2_m) hold.

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Lemma

Player B has a winning strategy in $Ssp(X)$.

Let F be a finite subset F of C. Then the set $F \times D$ is closed an nowhere dense in M, so $(U \cap M) \setminus (F \times D) \neq \emptyset$ for every non-empty open subset U of X.

Using this observation, for every partial play

$$(U_1, x_1, U_2, x_2, \ldots, U_n),$$

we can select a point

$$\tau(U_1, x_1, U_2, x_2, \ldots, U_n) \in (U_n \cap M) \setminus (\pi(\{x_1, \ldots, x_{n-1}\}) \times D),$$

where $\pi : M = C \times D \to C$ is the projection on the first coordinate. This defines a strategy τ for Player B.

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where $\pi : M = C \times D \to C$ is the projection on the first coordinate. This defines a strategy τ for Player B.

To show that this strategy is winning for B, let σ be an arbitrary strategy for Player A in $Ssp(X)$. Let $(U_1, x_1, U_2, x_2, \ldots)$ be the play produced by σ and τ; that is, $U_1 = \sigma(\emptyset)$, $x_1 = \tau(U_1)$,

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for $n \geq 2$.

For every $n \in \mathbb{N}$, let $x_n = (c_n, d_n) \in C \times D = M$.

By the choice of the stretegy τ,

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for $n \geq 2$. Therefore, the sequence $\{c_n : n \in \mathbb{N}\}$ is faithfully indexed; that is, $c_n \neq c_m$ for $n \neq m$.

Now the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence in X by the following lemma:

Lemma

If $x_n = (c_n, d_n) \in C \times D = M$ for every $n \in \mathbb{N}$ and $c_n \neq c_m$ whenever $m, n \in \mathbb{N}$ and $m \neq n$, then the sequence $\{x_n : n \in \mathbb{N}\}$ has a subsequence converging in X.

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Lemma

If $x_n = (c_n, d_n) \in C \times D = M$ *for every* $n \in \mathbb{N}$ *and* $c_n \neq c_m$ *whenever* $m, n \in \mathbb{N}$ *and* $m \neq n$*, then the sequence* $\{x_n : n \in \mathbb{N}\}$ *has a subsequence converging in* X*.*

We consider two cases.

Case 1. There exists $d \in D$ *such that* $N_d = \{n \in \mathbb{N} : d_n = d\}$ *is infinite.* Since $\{c_n : n \in N_d\}$ is an infinite sequence of elements of the Cantor set C, there exists an infinite set $K \subseteq N_d$ such that the sequence $\{c_n : n \in K\}$ converges to some $c \in C$. Now the subsequence $\{x_n : n \in K\}$ of the sequence $\{x_n : n \in \mathbb{N}\}$ converges to the point $(c, d) \in C \times D = M$.

Case 2. The set $N_d = \{n \in \mathbb{N} : d_n = d\}$ is finite for each $d \in D$. In this case, we can choose an infinite set $N \subseteq \mathbb{N}$ such that $d_m \neq d_n$ whenever $m, n \in N$ and $d_m \neq d_n$. Since C is compact metric, there is an infinite subset $J \subseteq N$ such that the sequence $\{c_n : n \in J\}$ converges to some point $c \in C$. Let $j : \mathbb{N} \to J$ and $k : \mathbb{N} \to \mathbb{N}$ be strictly increasing functions such that $c_{j(m)} \in V^c_{k(m)}$ for every $m \in \mathbb{N}$. Since $j(\mathbb{N}) \subseteq J \subseteq N$ and j is injective, the set

$$S = \{d_{j(m)} : m \in \mathbb{N}\} \tag{10}$$

is faithfully indexed. In particuar, S is a countably infinite subset of D. By the property (iii) of the family $\mathscr{H}$, we can find $h \in \mathscr{H}$ such that $S \cap h(\mathbb{N})$ is infinite. Therefore, the set

$$Q = \{l \in \mathbb{N} : d_{j(l)} \in h(\mathbb{N})\}$$

is infinite. Since k is an injection, the set $P = k(Q)$ is infinite as well. Define $p : P \to \mathbb{N}$ by

$$p(m) = h^{-1}(d_{j \circ k^{-1}(m)}) \ \text{ for } \ m \in P. \tag{11}$$

Case 2. The set $N_d = \{n \in \mathbb{N} : d_n = d\}$ *is finite for each* $d \in D$.
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Case 2. The set $N_d = \{n \in \mathbb{N} : d_n = d\}$ is finite for each $d \in D$. In this case, we can choose an infinite set $N \subseteq \mathbb{N}$ such that $d_m \neq d_n$ whenever $m, n \in N$ and $d_m \neq d_n$. Since C is compact metric, there is an infinite subset $J \subseteq N$ such that the sequence $\{c_n : n \in J\}$ converges to some point $c \in C$. Let $j : \mathbb{N} \to J$ and $k : \mathbb{N} \to \mathbb{N}$ be strictly increasing functions such that $c_{j(m)} \in V^c_{k(m)}$ for every $m \in \mathbb{N}$. Since $j(\mathbb{N}) \subseteq J \subseteq N$ and j is injective, the set

$$S = \{d_{j(m)} : m \in \mathbb{N}\} \tag{10}$$

is faithfully indexed. In particuar, S is a countably infinite subset of D. By the property (iii) of the family $\mathscr{H}$, we can find $h \in \mathscr{H}$ such that $S \cap h(\mathbb{N})$ is infinite. Therefore, the set

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Since P is infinite, p is an infinite partial function. Since $\mathscr{F}$ is a van Douwen MAD family, there exists $g \in \mathscr{F}$ such that the set

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$$h \circ g(k(m)) = h(g(k(m))) = h(p(k(m))) = h(h^{-1}(d_{j \circ k^{-1}(k(m))})) = d_{j(m)} \tag{13}$$

Since $c_{j(m)} \in V^c_{k(m)}$ for every $m \in \mathbb{N}$, from (4) and (13) we get

$$x_{j(m)} = (c_{j(m)}, d_{j(m)}) \in V^c_{k(m)} \times \{h \circ g(k(m))\} = M^{k(m)}_{c,g,h} \text{ for every } m \in T. \tag{14}$$

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Since the enumeration of $S = \{d_{j(m)} : m \in \mathbb{N}\}$ is faithful, so is the enumeration of the set $E = \{d_{j \circ k^{-1}\restriction_T(m)} : m \in T\} \subseteq h(\mathbb{N})$.
Since h is an injection by item (i) of the definition of $\mathscr{H}$, the map $h^{-1}\restriction_{h(\mathbb{N})}$ is injective. Now it follows from (11) and (12) that $g\restriction_T = p\restriction_T$ is an injection. Since T is infinite, $T \in \mathcal{I}_g$ by (2). In particular, $\mathcal{I}_g \neq \emptyset$, so $g \in \mathscr{G}$ by the definition of $\mathscr{G}$. Since $T \in \mathcal{I}_g$, we can use item (b) of the definition of $\mathcal{A}_g$ to find $A \in \mathcal{A}_g$ such that $A \cap T$ is infinite.
It remains only to observe that the sequence $\{x_{j(m)} : m \in A \cap T\}$ converges to the point (c, g, h, A) in X. Indeed, let $n \in \mathbb{N}$ be arbitrary. Since the function k is monotonically increasing, there exists $l \in \mathbb{N}$ such that $k(m) > n$ provided that $m \geq l$. It follows from (5), (9) and (14) that $x_{j(m)} \in O^n_{c,g,f,A} \subseteq B^n_{c,g,f,A}$ for $m \in A \cap T$ satisfying $m \geq l$.

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Happy Birthday, Marion!