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# Ideal modification of Arkhangel'skiĭ´s $\alpha_4$ property

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## Definition (Ideal)

The family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called **ideal**, if it has properties:

(I1) $A \in \mathcal{I},\ B \subseteq A \to B \in \mathcal{I}$,

(I2) $A, B \in \mathcal{I} \to A \cup B \in \mathcal{I}$,

(I3) $\omega \notin \mathcal{I}$,

(I4) $(\forall n \in \omega)\ \{n\} \in \mathcal{I}$.

- **The Frechét ideal**, denoted as Fin, is a set $[\omega]^{<\aleph_0}$.

# Basic terms

## Definition (Ideal)

The family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called **ideal**, if it has properties:

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## Definition (Pseudounion)

A set $B$ is **pseudounion** of family $\mathcal{A}$, if $\omega \setminus B$ is infinite and if $A \subseteq^* B$ for any $A \in \mathcal{A}$.

- Ideal $\mathcal{I}$ is **tall** if for any $B \in [\omega]^\omega$, there exists an $A \in \mathcal{I}$ such that $A \cap B$ is infinite.
- Note that an ideal $\mathcal{I}$ has a pseudounion if and only if $\mathcal{I}$ is not tall.

# Arkhangel'skiĭ´s $\alpha_4$ property

## Definition (Classical property)

A space has **the property** $\alpha_4$ if there is for each $x$ and each sequence $\langle F_n : \ n \in \omega \rangle$ from $\Gamma_x$ , a $B$ in $\Gamma_x$ such that for infinitely many $n$, $B \cap F_n$ is nonempty.

- $\Gamma_x$ the set of $A \subset X \setminus \{x\}$ such that $A$ is countably infinite, and each neighborhood of $x$ contains all but finitely many elements of $A$.
- Note
  - Compact Fréchet spaces have the property $\alpha_4$.
  - Nogura[6]: There are two Fréchet compact spaces $X$ and $Y$ which both have the property $\alpha_4$ but the space $X \times Y$ has not the property $\alpha_4$.

# Ideal Convergence

## Definition (Point ideal convergence)

We say that a sequence $\langle x_n : n \in \omega \rangle$ of elements of a topological space $X$ is $\mathcal{I}$-**convergent to** $x \in X$ if the set $\{n \in \omega : x_n \notin U\} \in \mathcal{I}$ for each neighborhood $U$ of $x$.

- We write $x_n \xrightarrow{\mathcal{I}} x$.
- Now we consider real functions $f_n : X \to \mathbb{R}$.

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## Definition

We say that a sequence $\langle f_n : \ n \in \omega \rangle$ $\mathcal{I}$-convergent to a function $f$ on $X$ if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for every $x \in X$.

- We write $f_n \xrightarrow{\mathcal{I}} f$
- Point $\mathcal{I}$-convergence

**Definition ($\mathcal{I}$-quasi-normal convergence)**

The sequence $\langle f_n : n \in \omega \rangle$ is called $\mathcal{I}$-**quasi-normal convergent** to $f$ on $X$ if there exists a sequence of positive reals $\langle \varepsilon_n : n \in \omega \rangle$ and $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ such that $\{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$, denoted $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} 0$.

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- $\langle \varepsilon_n : n \in \omega \rangle$ is called control sequence
- especially, if control sequence is $\langle 2^{-n} : n \in \omega \rangle$ we say about **strongly** $\mathcal{I}$-**quasi normal convergence** of $f_n$ to $f$, written $f_n \xrightarrow{\mathrm{s}\mathcal{I}\mathrm{QN}} f$.

**Definition ($(\mathcal{I},\mathcal{J})$wQN-space)**

A topological space $X$ is an $(\mathcal{I},\mathcal{J})\mathrm{wQN}$-**space** if for any sequence $\langle f_n : n \in \omega\rangle$ of continuous real functions $\mathcal{I}$-converging to 0 on $X$, there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.

**Definition ($(\mathcal{I},\mathrm{s}\mathcal{J})$wQN-space)**

A topological space $X$ is called an $(\mathcal{I},\mathbf{s}\mathcal{J})\mathrm{wQN}$-**space** , if there is a sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive reals such that $\varepsilon_n \to 0$ and for any sequence $\langle f_n : n \in \omega\rangle$ of continuous functions on $X$ converging to zero there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control sequence $\langle \varepsilon_n : n \in \omega\rangle$.

# Arkhangel'skiĭ´s $\alpha_4$ property

- The $C_p(X)$ - a set of continuous functions from $X$ to $\mathbb{R}$ endowed with the Tychonoff product topology.

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## Definition (Ideal modification)

The $C_p(X)$ has **the property** $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if:

for any sequence $\langle\langle f_{n,m} : \ m \in \omega\rangle : \ n \in \omega\rangle$ of sequences of continuous real functions such that $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each $n$, there exists a sequence $\langle m_n : \ n \in \omega\rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

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- $(\mathrm{Fin}, \mathrm{Fin}\text{-}\alpha_4)$ is the same as the classical property
- D. Fremlin and M. Scheepers [3, 6, 7] showed:
  Any topological space $X$ is a $\mathrm{wQN}$-space if and only if $C_p(X)$ has a property $(\mathrm{Fin}, \mathrm{Fin}\text{-}\alpha_4)$
- L. Bukovský, P. Das and J. Šupina[1] proved ideal version:
  Any topological space $X$ is an $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space if and only if $C_p(X)$ has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$.

# Monotone modification

Suppose that $\mathcal{J}$ is an ideal on $\omega$.

## Definition

The $C_p(X)$ has **the property** $\mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$ if:

for any sequence $\langle\langle f_{n,m} : \ m \in \omega\rangle : \ n \in \omega\rangle$ of monotone sequences of continuous real functions such that $f_{n,m} \to 0$ for each $n$, there exists a sequence $\langle m_n : \ n \in \omega\rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

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## Proposition

*Let $X$ be a topological space and $\mathcal{J}$ is an ideal on $X$ with a pseudounion. Then the equivalence* $\mathrm{mon}(\mathrm{Fin}\text{-}\alpha_4) \Leftrightarrow \mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$ *holds.*

- Similarly, since any monotone sequence on a compact topological space convergent to $0$ is uniformly convergent, by additivity properties of $\mathrm{mon}(\mathrm{Fin}\text{-}\alpha_4)$ we can say: any $\sigma$-compact space has $\mathrm{mon}(\mathrm{Fin}\text{-}\alpha_4)$ and thus $\mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$.

# Monotone modification

**Definition (Almost monotone sequence)**

We say that a sequence $\langle f_n : \ n \in \omega \rangle$ is $\mathcal{I}$-**almost monotone sequence** if $\{n; \ f_n \not\leq f_m\} \in \mathcal{I}$ for every $m \in \omega$.

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## Lemma (Upper bounding by monotone sequence)

*Let $X$ be a set. For an almost monotone sequence $\langle f_n : \ n \in \omega \rangle$ such that $f_n \to 0$ there is a monotone sequence $\langle g_n : \ n \in \omega \rangle$ such that $g_n \to 0$ and $f_n \leq g_n$ for any $n \in \omega$.*

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## Definition

The $C_p(X)$ has **the property** $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if:

for any sequence $\langle \langle f_{n,m} : m \in \omega \rangle : n \in \omega \rangle$ of $\mathcal{I}$-almost monotone sequences of continuous real functions such that $f_{n,m} \to 0$ for each $n$, there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

**Lemma**

*Let $X$ be a topological space. Then $C_p(X)$ has the property $m(\mathcal{J}\text{-}\alpha_4)$ if and only if $C_p(X)$ has the property $mon(\mathcal{J}\text{-}\alpha_4)$.*

- $\mathrm{mon}(\mathcal{J}\text{-}\alpha_4) \to \mathrm{m}(\mathcal{J}\text{-}\alpha_4)$ - trivial because $\mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$ satisfies $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$,
- $\mathrm{m}(\mathcal{J}\text{-}\alpha_4) \to \mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$ - using previous Lemma.

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## Lemma

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- $\mathrm{m}(\mathcal{J}\text{-}\alpha_4) \to \mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$ - using previous Lemma.

## Definition (modification of the monotonic quasi-normal space)

A topological space $X$ is called an $\mathrm{s}\mathcal{J}\mathrm{wmQN}$-**space**, if there is a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals such that $\varepsilon_n \to 0$ and for any monotone sequence $\langle f_n : n \in \omega \rangle$ of continuous functions on $X$ converging to zero there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control sequence $\langle \varepsilon_n : n \in \omega \rangle$.

# Monotone modification

**Theorem**

*If $X$ is a perfectly normal topological space then the following are equivalent. Moreover, if $X$ is arbitrary topological space then* $(\mathrm{a}) \equiv (\mathrm{b})$.

(a) $X$ *is an* $\mathrm{s}\mathcal{J}\mathrm{wmQN}$*-space.*

(b) $C_p(X)$ *has the property* $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$.

(c) $X$ *possesses a* $\mathcal{J}$*-Hurewicz property.*

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(c) *$X$ possesses a $\mathcal{J}$-Hurewicz property.*

**Definition ($\mathcal{J}$-Hurewicz property)**

A topological space $X$ has $\mathcal{J}$-**Hurewicz property** if for each sequence $\langle \mathcal{U}_n : \ n \in \omega \rangle$ of open covers of $X$ there are finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \omega$ such that for each $x \in X$, $\{n \in \omega, \ x \notin \bigcup \mathcal{V}_n\} \in \mathcal{J}$.

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- Chandra [4] showed:
  If $X$ is a topological space with the $\mathcal{J}$-Hurewicz property, then $C_p(X)$ has $\mathrm{mon}(\mathcal{J}\text{-}\alpha_4)$.

# Monotone modification

**Theorem**

*If $X$ is a perfectly normal topological space then the following are equivalent. Moreover, if $X$ is arbitrary topological space then $\mathrm{(a)} \equiv \mathrm{(b)}$.*

(a) $X$ *is an* $\mathrm{s}\mathcal{J}\mathrm{wmQN}$*-space.*

(b) $C_p(X)$ *has the property* $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$*.*

(c) $X$ *possesses a* $\mathcal{J}$*-Hurewicz property.*

- $\mathrm{(c)} \to \mathrm{(b)}$ has been shown by D. Chandra [4].
- $\mathrm{(b)} \to \mathrm{(a)}$ we prove the similar way as proof by Bukovský-Haleš of [2]
- $\mathrm{(a)} \to \mathrm{(c)}$ is based on proof of Lemma 3 by M. Scheepers [5]

# Conclusion

For any ideals $\mathcal{I}, \mathcal{J}$ on $\omega$ we have

$$\text{mon}(\mathcal{J}\text{-}\alpha_4) \equiv \text{m}(\mathcal{J}\text{-}\alpha_4) \to \text{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$$

$$(\mathcal{I}, \mathcal{J}\text{-}\alpha_4) \to \text{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$$

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$$(\mathcal{I}, \mathcal{J}\text{-}\alpha_4) \to \mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$$

## What else?

- Which other spaces apart from monotone spaces have the property $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$?
- Which situations are the properties $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ and $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ equal in?
- What else we could obtain for $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$?
- ...

Thank you for your attention

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