Ideals and three sequence selection principles

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All spaces are assumed to be Hausdorff and infinite.

Diagrams hold for perfectly normal space.

All covers are assumed to be countable.

Scheepers M., *Combinatorics of open covers I: Ramsey theory,* Topology Appl. **69** (1996), 31–62.

$\mathsf{S}_1(\Gamma,\Gamma)$-property

X possesses the property $\mathsf{S}_1(\Gamma,\Gamma)$ if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open γ-covers there exist sets $U_n \in \mathcal{U}_n$ such that $\langle U_n;\ n \in \omega\rangle$ is a γ-cover.

- γ-cover $\langle U_n;\ n \in \omega\rangle$ - every $x \in X$ lies in all but finitely many members of $\langle U_n;\ n \in \omega\rangle$, $\Gamma(X)$ or Γ denote the family of all countable open γ -covers

Just W., Miller A.W., Scheepers M. and Szeptycki P.J., *Combinatorics of open covers* II, Topology Appl. **73** (1996), 241–266.

Scheepers M., *A sequential property of $C_p(X)$ and a covering property of Hurewicz,* Proc. Amer. Math. Soc. **125** (1997), 2789–2795.

- Tychonoff $\mathsf{S}_1(\Gamma,\Gamma)$-space is zero-dimensional
- any $\mathsf{S}_1(\Gamma,\Gamma)$-subset of a metric separable space is perfectly meager
- perfectly normal $\mathsf{S}_1(\Gamma,\Gamma)$-space has Hurewicz property
- non$(\mathsf{S}_1(\Gamma,\Gamma)) = \mathfrak{b}$
- $\mathfrak{h} \leq$ add$(\mathsf{S}_1(\Gamma,\Gamma)) \leq \mathfrak{b}$
- any γ-set is an $\mathsf{S}_1(\Gamma,\Gamma)$-space
- $\mathfrak{b}$-Sierpiński set is an $\mathsf{S}_1(\Gamma,\Gamma)$-space (exists under $\mathfrak{b} =$ cov$(\mathcal{N}) =$ cof$(\mathcal{N})$)

Recław I., *Metric spaces not distinguishing pointwise and quasinormal convergence of real functions,* Bull. Acad. Polon. Sci. **45** (1997), 287–289.

- there exists an uncountable $\mathsf{S}_1(\Gamma,\Gamma)$-space (of cardinality $\mathfrak{b}$)

Sequence Selection Principles

Архангельский А.В. (Arkhangel'skiĭ A.V.), *Спектр частот топологического пространства и классификация пространств,* ДАН СССР, **206**:2 (1972), 265–268. English translation *The frequency spectrum of a topological space and the classification of spaces,* Soviet Math. Dokl. **13** (1972), 1185–1189.

A topological space Y is (α_4)-space if for any sequence $\langle S_n : n \in \omega \rangle$ of sequences converging to a point $y \in Y$, there exists a sequence S converging to y such that $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

Fremlin D.H., *Sequential convergence in* $C_p(X)$, Comment. Math. Univ. Carolin. **35** (1994), 371–382.

Scheepers M., *A sequential property of* $C_p(X)$ *and a covering property of Hurewicz,* Proc. Amer. Math. Soc. **125** (1997), 2789–2795.

A topological space Y has sequence selection property, if for any $x \in Y$ and for any sequence $\langle S_n : n \in \omega \rangle$ of sequences converging to x there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \to x$ and $x_n \in S_n$ for each $n \in \omega$.

A topological space $C_p(X)$ has monotonic sequence selection property, if for any sequence $\langle S_n : n \in \omega \rangle$ of sequences in $C_p(X)$ converging to zero monotonically there is a sequence $\{f_n\}_{n=0}^{\infty}$ such that $f_n \to 0$ and $f_n \in S_n$ for each $n \in \omega$.

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perfectly normal space X

L. Bukovský [2008] M. Sakai [2009]		M. Scheepers [1999] D.H. Fremlin [2003] L. Bukovský and J. Haleš [2007]		M. Scheepers [1999] L. Bukovský, I. Reclaw and M. Repický [2001] L. Bukovský and J. Haleš [2003]
X possesses SSP^*.		$\mathrm{C}_p(X)$ possesses (α_4).		$\mathrm{C}_p(X)$ possesses MSSP.
X is a wQN*-space.	→	X is a wQN-space.	→	X is an mQN-space.
X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.		X is an $\mathrm{S}_1(\Gamma_{\mathrm{sh}},\Gamma)$-space.		X has $\mathcal{U}_{fin}(\mathcal{O}_\omega,\Gamma)$.

$\mathfrak{b}$-Sierpiński set, γ-set

compact set

X is perfectly meager

X is zero-dimensional

X has count. Menger property

↑

Scheepers Conjecture

A family $\mathcal{K} \subseteq \mathcal{P}(\omega)$ is called an ideal if

a) $B \in \mathcal{K}$ for any $B \subseteq A \in \mathcal{K}$,
b) $A \cup B \in \mathcal{K}$ for any $A, B \in \mathcal{K}$,
c) $\mathrm{Fin} = [\omega]^{<\aleph_0} \subseteq \mathcal{K}$,
d) $\omega \notin \mathcal{K}$.

$\mathcal{I}$, $\mathcal{J}$, $\mathcal{S}$, $\mathcal{K}$ are ideals in the following.

Das P., *Certain types of open covers and selection principles using ideals,* Houston J. Math. **39** (2013), 637–650.

Di Maio G., Kočinac Lj.D.R., *Statistical convergence in topology,* Topology Appl., **156** (2008), 28–45.

A sequence $\langle U_n : n \in \omega \rangle$ of subsets of a topological space X is said to be an $\mathcal{I}$**-**γ**-cover**, if for every n, $U_n \neq X$, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$.

$\mathcal{I}$-Γ - the family of all open $\mathcal{I}$-γ-covers

$\mathrm{S}_1(\Gamma, \mathcal{J}\text{-}\Gamma)$-space

$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space

Bukovský L., Das P. and Šupina J., *Ideal quasi-normal convergence and related notions*, Colloq. Math. 146 (2017), 265-281.

Any γ-set is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

Observation (V. Šottová–J.Š.)

Any $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$*-space is an* $S_1(\Gamma, \Omega)$*-space.*

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A sequence $\langle U_n : n \in \omega \rangle$ of subsets of a topological space X is said to be an **$\mathcal{I}$-γ-cover**, if for every n, $U_n \neq X$, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$.

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Any $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space is an $S_1(\Gamma, \Omega)$-space.

$$
\begin{array}{ccccc}
\mathrm{S}_1(\Omega,\Gamma) & & & & \\
\downarrow & & & & \\
\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma) & & \\
\downarrow & & \downarrow & & \\
\mathrm{S}_1(\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1(\Gamma,\mathcal{J}\text{-}\Gamma) & \longrightarrow & \mathrm{S}_1(\Gamma,\Omega)
\end{array}
$$

Lemma (V. Šottová–J.Š.)

For any countable ω-cover $\mathcal{U}$ and its bijective enumeration $\langle U_n : n \in \omega \rangle$ there is an ideal $\mathcal{I}$ such that $\langle U_n : n \in \omega \rangle$ is an $\mathcal{I}$-γ-cover.

Theorem (V. Šottová–J.Š.)

A topological space X is an $\mathrm{S}_1(\Omega,\Gamma)$-space if and only if for every ideal $\mathcal{I}$, X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\Gamma)$-space.

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Lemma (V. Šottová–J.Š.)

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A topological space X is an $S_1(\Omega,\Gamma)$-space if and only if for every ideal $\mathcal{I}$, X is an $S_1(\mathcal{I}\text{-}\Gamma,\Gamma)$-space.

$\mathcal{I}$-convergence $x_n \xrightarrow{\mathcal{I}} x$

$$(\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |x_n - x| < \varepsilon)$$

$\mathcal{I}$-pointwise convergence $f_n \xrightarrow{\mathcal{I}} f$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |f_n(x) - f(x)| < \varepsilon)$$

Das P. and Chandra D., *Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence of real functions*, Comment. Math. Univ. Carolin. **54** (2013), 83–96.

$\mathcal{I}$-quasi-normal convergence $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$

there exists $\{\varepsilon_n\}_{n=0}^{\infty}$ $\mathcal{I}$-converging to 0 such that

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Filipów R. and Staniszewski M., *On ideal equal convergence*, Cent. Eur. J. Math **12** (2014), 896–910.

($\mathcal{I}$,$\mathcal{J}$)-equal convergence $f_n \xrightarrow{(\mathcal{I},\mathcal{J})\text{-e}} 0$

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The $\mathrm{C}_p(X)$ has the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if for any sequence $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of sequences of continuous real functions such that $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n, there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

Theorem

If X is a normal topological space then the following are equivalent.

(a) X *is an* $(\mathcal{I}, s\mathcal{J})\mathrm{wQN}$*-space.*

(b) $\mathrm{C}_p(X)$ *has the property* $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$*.*

(c) X *is an* $S_1(\mathcal{I}\text{-}\Gamma^{sh}, \mathcal{J}\text{-}\Gamma)$*-space.*

$(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space

X is called an $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space, if there is a sequence $\langle\varepsilon_n : n \in \omega\rangle$ of positive reals such that $\varepsilon_n \to 0$ and for any sequence $\langle f_n : n \in \omega\rangle$ of continuous functions on X converging to zero there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control sequence $\langle\varepsilon_n : n \in \omega\rangle$.

i.e. $f_{m_n} \xrightarrow{(\mathcal{J},\mathrm{Fin})\text{-}\mathrm{e}} 0$ with control sequence $\langle\varepsilon_n : n \in \omega\rangle$ in the terminology of R. Filipów and M. Staniszewski.

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$(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space with control sequence $\langle \varepsilon_n : n \in \omega \rangle$

X is called an $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space with control sequence $\langle \varepsilon_n : n \in \omega \rangle$, if for any sequence $\langle f_n : n \in \omega \rangle$ of continuous real functions converging to 0 via ideal $\mathcal{I}$ on X, there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control sequence $\langle \varepsilon_n : n \in \omega \rangle$.

Theorem (V. Šottová–J.Š.)

Let X be a topological space. X is an $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space with control sequence $\langle \delta_n : n \in \omega \rangle$ if and only if X is an $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space.

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Theorem (V. Šottová–J.Š.)

Let X be a topological space. X is an $(\mathcal{I}, s\mathcal{J})\text{wQN}$-space with control sequence $\langle \delta_n : n \in \omega \rangle$ if and only if X is an $(\mathcal{I}, s\mathcal{J})\text{wQN}$-space.

A sequence $\langle f_n : n \in \omega \rangle$ is called $\mathcal{I}$-almost monotone sequence if $\{n;\ f_n \not\leq f_m\} \in \mathcal{I}$ for every $m \in \omega$.

$\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$

The $\mathrm{C}_p(X)$ has the property $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if for any sequence $\langle \langle f_{n,m} : m \in \omega \rangle : n \in \omega \rangle$ of $\mathcal{I}$-almost monotone sequences of continuous real functions such that $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n, there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

Chandra D., *Some remarks on sequence selection properties using ideals*, Mat. Vesnik **68** (2016), 3944.

If X possesses a $\mathcal{J}$-Hurewicz property then $\mathrm{C}_p(X)$ has the property $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$.

Theorem (V. Šottová–J.Š.)

If X is a perfectly normal topological space then the following are equivalent.

(a) X *is an* $\mathrm{s}\mathcal{J}\mathrm{wmQN}$*-space.*

(b) $\mathrm{C}_p(X)$ *has the property* $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$*.*

(c) X *possesses a* $\mathcal{J}$*-Hurewicz property.*

A sequence $\langle f_n : n \in \omega \rangle$ is called $\mathcal{I}$-almost monotone sequence if $\{n;\ f_n \nleq f_m\} \in \mathcal{I}$ for every $m \in \omega$.

$\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$

The $\mathrm{C}_p(X)$ has the property $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if for any sequence $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of $\mathcal{I}$-almost monotone sequences of continuous real functions such that $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n, there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

Chandra D., *Some remarks on sequence selection properties using ideals*, Mat. Vesnik **68** (2016), 3944.

If X possesses a $\mathcal{J}$-Hurewicz property then $\mathrm{C}_p(X)$ has the property $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$.

Theorem (V. Šottová–J.Š.)

If X is a perfectly normal topological space then the following are equivalent.

(a) X *is an* $\mathrm{s}\mathcal{J}\mathrm{wmQN}$*-space.*

(b) $\mathrm{C}_p(X)$ *has the property* $\mathrm{m}(\mathcal{J}\text{-}\alpha_4)$.

(c) X *possesses a* $\mathcal{J}$*-Hurewicz property.*

A sequence $\langle f_n : n \in \omega \rangle$ is called $\mathcal{I}$-almost monotone sequence if $\{n;\ f_n \not\leq f_m\} \in \mathcal{I}$ for every $m \in \omega$.

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Šupina J., *Ideal* QN*-spaces*, J. Math. Anal. Appl. 434 (2016) 477-491.

$$\lambda(\mathcal{I},\mathcal{J}) = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq {}^{\omega}\mathcal{I} \wedge (\forall \varphi \in {}^{\omega}\omega)(\exists \langle B_n : n \in \omega\rangle \in \mathcal{A})\ \{n;\ \varphi(n) \in B_n\} \in \mathcal{J}^{+}\}$$

$$\mathfrak{p} \le \lambda(\mathcal{I},\mathrm{Fin}) \le \substack{\lambda(\mathcal{I},\mathcal{J}) \\ \mathfrak{b}} \le \lambda(\mathrm{Fin},\mathcal{J}) = \mathfrak{b}_{\mathcal{J}} \le \mathfrak{d}$$

Lemma

Let X be a topological space. If $|X| < \lambda(\mathcal{I},\mathcal{J})$ then X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

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Let D be a discrete topological space. Then the following statements are equivalent.

(a) *D is an $S_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.*
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(d) *$\mathrm{C}_p(D)$ has the property $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$.*
(e) *$|D| < \lambda(\mathcal{I},\mathcal{J})$.*

$$\mathrm{non}((\mathcal{I},\mathcal{J}\text{-}\alpha_4)) = \mathrm{non}(\mathrm{m}(\mathcal{I},\mathcal{J}\text{-}\alpha_4)) = \mathrm{non}((\mathcal{I},\mathrm{s}\mathcal{J})\mathrm{wQN}\text{-space}) = \lambda(\mathcal{I},\mathcal{J})$$

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Šupina J., *Ideal* QN-*spaces*, J. Math. Anal. Appl. 434 (2016) 477-491.

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If $\mathrm{C}_p(X)$ has the property $\mathrm{m}(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$, then X is bounded in $({}^\omega\omega, \leq^{\mathcal{J}})$.

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da Silva S.G., *The $\mathcal{I}$-Hurewicz property and bounded families modulo an ideal*, Topology Appl., (2017), preprint.

If X possesses a $\mathcal{J}$-Hurewicz property, then X is bounded in $({}^\omega\omega, \leq^{\mathcal{J}})$.

Scheepers' Conjecture

Scheepers M., *Sequential convergence in $C_p(X)$ and a covering property,* East-West J. of Mathematics **1** (1999), 207–214.

Scheepers' Conjecture

Any perfectly normal wQN-space is an $S_1(\Gamma, \Gamma)$-space.

Sakai M., *The Ramsey property for $C_p(X)$*, Acta Math. Hungar. **128** (2010), 96–105.

There is non-normal QN-space which is not an $S_1(\Gamma, \Gamma)$-space.

Dow A., *Two classes of Fréchet-Urysohn spaces,* Proc. Amer. Math. Soc. **131** (1990), 241–247.

Laver R., *On the consistency of Borel's conjecture,* Acta Math. **137** (1976), 151–169.

Tsaban B. and Zdomskyy L., *Hereditary Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations,* J. Eur. Math. Soc. (JEMS), **14** (2012), 353–372.

The theory

$$\textbf{ZFC} + \text{"Scheepers' conjecture holds"}$$

is consistent relatively to **ZFC**.

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The theory

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Ohta–Sakai's properties

Scheepers' Conjecture

Any perfectly normal wQN-space is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.

Ohta H. and Sakai M., *Sequences of semicontinuous functions accompanying continuous functions*, Topology Appl. **156** (2009), 2683-2906.

USC	X has property USC, if whenever $\langle f_n : n \in \omega\rangle$ of **upper** semicontinuous functions with values in $[0,1]$ converges to zero, there is $\langle g_n : n \in \omega\rangle$ of continuous functions converging to zero such that $f_n \leq g_n$ for any $n \in \omega$.
USC_s	X has property USC_s, if whenever $\langle f_n : n \in \omega\rangle$ of upper semicontinuous functions with values in $[0,1]$ converges to zero, there is $\langle g_n : n \in \omega\rangle$ of continuous functions converging to zero and an increasing sequence $\{n_m\}_{m=0}^{\infty}$ such that $f_{n_m} \leq g_m$ for any $m \in \omega$.

Proposition

Any wQN-space with USC_s is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.

Theorem

Every separable metrizable space with USC_s is perfectly meager.

Scheepers' Conjecture

Any perfectly normal wQN-space is an $\mathrm{S}_1(\Gamma, \Gamma)$-space.

J. Haleš [2005], M. Sakai [2007], L. Bukovský and J. Haleš [2007]

Theorem (H. Ohta – M. Sakai)

Let X be a perfectly normal space with $\mathrm{Ind}(\mathrm{X}) = 0$.

(1)	*X possesses USC.*	$(1)^s$	*X possesses USC_s.*
(2)	*X is (γ, γ)-shrinkable.*	$(2)^s$	*Open γ-cover of X is shrinkable.*
(3)	*X is a σ-set.*	$(3)^s$	*X is a $\gamma\gamma_{co}$-space.*

Šupina J., *On Ohta–Sakai's properties of a topological space,* Topology Appl. **190** (2015), 119–134.

Theorem

A topological space X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space if and only if X is a wQN-space with the property $\mathrm{wED}(\widetilde{\mathcal{U}}, \mathrm{C}_p(X))$.

A topological space X is a σ-set if every F_σ subset of X is a G_δ set in X.

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Das P. and Chandra D., *Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence of real functions,* Comment. Math. Univ. Carolin. **54** (2013), 83–96.

Das P. and Chandra D., *($\mathcal{I}$,$\mathcal{J}$)-quasinormal spaces*, manuscript.

X is called an $(\mathcal{I},\mathcal{J})\mathrm{wQN}$-space, if for any sequence $\langle f_n : n \in \omega\rangle$ of continuous functions on X converging to zero there exists a sequence $\langle m_n : n \in \omega\rangle$ and a sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive reals $\mathcal{J}$-converging to zero such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control sequence $\langle \varepsilon_n : n \in \omega\rangle$.

Šupina J., *Ideal QN-spaces,* J. Math. Anal. Appl. **435** (2016), 477–491.

Theorem

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals such that $\mathcal{J}$ is not a weak $\mathrm{P}(\mathcal{I})$-ideal. Then there is an $(\mathcal{I},\mathcal{J})\mathrm{wQN}$-space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

Theorem

Let $\mathcal{J} \subseteq \mathcal{P}(\omega)$ be an ideal. If $\mathrm{Fin} \times \mathrm{Fin} \leq_K \mathcal{J}$ then there is a $\mathcal{J}\mathrm{wQN}$-space $X \subseteq \mathbb{R}$ which is not an $\mathrm{S}_1(\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

Bukovský L., *On* wQN_* *and* wQN^* *spaces*, Topology Appl. **156** (2008), 24-27.

Bukovský L. and Haleš J., *On Hurewicz properties,* Topology Appl. **132** (2003), 71–79.

Bukovský L. and Haleš J., QN*-spaces,* wQN*-spaces and covering properties,* Topology Appl. **154** (2007), 848–858.

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Fremlin D.H., SSP *and* wQN*,* Notes of 14.01.2003, http://www.essex.ac.uk/maths/staff/fremlin/preprints.htm.

Haleš J., *On Scheepers' conjecture,* Acta Univ. Carolinae Math. Phys. **46** (2005), 27–31.

Sakai M., *The sequence selection properties of* $\mathrm{C}_p(X)$*,* Topology Appl. **154** (2007), 552–560.

Sakai M., *Selection principles and upper semicontinuous functions*, Colloq. Math. **117** (2009), 251-256.

Scheepers M., *Sequential convergence in* $\mathrm{C}_p(X)$ *and a covering property,* East-West J. of Mathematics **1** (1999), 207–214.

Thanks for Your attention!