Definable Versions of Menger's Conjecture

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The following pattern has been studied by analysts for more than a hundred years:
Simply defined sets of reals have a nice property, e.g. Borel sets are measurable, but, using the Axiom of Choice, we can construct a complicated non-measurable set. Set theorists have studied more complicated but yet "constructive" sets of reals and found they may or may not be measurable, depending on what set-theoretic axioms one assumed. We shall perform a similar investigation in the field of *Selection Principles*, culminating in an exact calibration of consistency strength.
Portions of this work are joint with S. Todorcevic and S. Tokgöz.

Definition 1

A space is **Menger** if whenever $\{\mathcal{U}_n\}_{n<\omega}$ is a sequence of open covers, there exist finite $\{\mathcal{V}_n\}_{n<\omega}$ such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n : n < \omega\}$ is a cover.

Definition 2

A space is **Hurewicz** if every Čech-complete space including it includes a σ-compact subspace including it. (A space is Čech-complete if it is a G_δ in every compactification. For metrizable spaces, Čech-completeness is the same as completeness.) Equivalently, whenever $\{\mathcal{U}_n\}_{n<\omega}$ are open covers of X, none including a finite subcover, there is a decomposition $X = \bigcup_{k<\omega} X_k$, such that for each k, there are finite $\mathcal{F}_1^k \subseteq \mathcal{U}_1$, $\mathcal{F}_2^k \subseteq \mathcal{U}_2, \ldots$, such that for each $x \in X_k$, $x \in \bigcup \mathcal{F}_n^k$ for all but finitely many n.

Menger's Conjecture: Menger subsets of $\mathbb{R}$ are σ-compact.
Hurewicz's Conjecture: Hurewicz subsets of $\mathbb{R}$ are σ-compact.

Proposition 1 (Various)

σ-compact $\to$ Hurewicz $\to$ Menger. No arrow reverses, even for subsets of $\mathbb{R}$.

Proposition 2 (Hurewicz 1925)

Completely metrizable (indeed, analytic metrizable) Menger spaces are σ-compact.

Problem 1

Are "definable" Menger subsets of $\mathbb{R}$ σ-compact?

Proposition 3 (Miller-Fremlin 1988)

$V = L$ implies there is a co-analytic (complement of analytic) Menger subset of $\mathbb{R}$ which is not σ-compact.

Definition 3

The **projective** subsets of $\mathbb{R}$ are obtained by closing the Borel sets under continuous image and complementation.

Definition 4

Let $X \subseteq {}^{\omega}\omega$. In the game $G(X)$, player I picks $a_0 \in \omega$, player II picks $a_1 \in \omega$, player I picks $a_2 \in \omega$, etc. I wins iff $\{a_n\}_{n<\omega} \in X$. $G(X)$ is **determined** if either I or II has a **winning strategy**. The **Axiom of Projective (respectively, co-analytic) Determinacy** says *all projective (co-analytic)* games are determined.

Theorem 4 (Miller-Fremlin, Tall-Tokgöz)

PD *(***CD***) implies all projective (co-analytic) Menger subsets of* $\mathbb{R}$ *are* σ*-compact.*

Corollary 5 (TT)

If there is a measurable cardinal, then co-analytic Menger subsets of $\mathbb{R}$ *are* σ*-compact.*

It is known that **CD** is equiconsistent with the existence of every $a^\sharp$, for $a \subseteq \omega$.

Problem 2

Without large cardinals, is it consistent that co-analytic (projective?) Menger subsets of $\mathbb{R}$ *are* σ*-compact?*

Theorem 6 (T-Todorcevic-T)

If it is consistent there is an inaccessible cardinal, it is consistent that projective Menger subsets of $\mathbb{R}$ are σ-compact.

Proof.

Use a variation **OGA*(projective)** of Todorcevic's **Open Graph Axiom** (formerly known as the *Open Coloring Axiom*) mentioned in Feng (1993):

OGA*(projective) If $X \subseteq \mathbb{R}$ is uncountable projective and $[X]^2 = K_0 \cup K_1$ is a partition with K_0 open in the relative topology, then either there is a **perfect** $A \subseteq X$ with $[A]^2 \subseteq K_0$, or $X = \bigcup_{n<\omega} A_n$, with $[A_n]^2 \subseteq K_1$ for all $n < \omega$.

Remark

OGA*(projective) does not imply **OGA**.

Proposition 7 (Todorcevic in Feng 1993)

OGA*(projective) *is equiconsistent with an inaccessible cardinal.*

Theorem 6 is proved by showing:

OGA*(projective) $\rightarrow$ **HD(projective)** $\rightarrow$ Menger projective sets of reals are σ-compact.

Hurewicz Dichotomy for projective sets Let E be a compact metrizable space and let A, B be disjoint projective subsets of E. Either there is a σ-compact $C \subseteq E$ such that $A \subseteq C$ and $C \cap B = \emptyset$, or there is a copy F of the Cantor set such that $F \subseteq A \cup B$ and $F \cap B$ is countable dense in F. $\square$

Ideas of proof. Let A be projective, not σ-compact. WLOG $A \subseteq$ Cantor set K. $B = K \setminus A$. Let $\hat{A} = \bigcup_{a \in A} \hat{a}$, where

$$\hat{a} = \{c : c \text{ is an infinite chain in } \{0,1\}^{<\omega} \text{ such that } \bigcup c = a\}.$$

Let $X = \hat{A} \times \hat{B}$. Let
$K_0 = \{\{\{a, b\}, \{a', b'\} \in [X]^2$ such that $(a \cap b') \cup (b \cap a') \neq \emptyset\}$.
K_0 is open in X^2. A is not σ-compact implies X is not a countable union of K_1-homogeneous sets.. By OGA*(projective), there is a perfect $F \subseteq X$ such that $[F]^2 \subseteq K_0$. We associate perfect sets to pruned trees and use the perfect tree associated with F and the fact that A is not σ-compact to recursively construct a superperfect (infinite splitting) tree such that its chains form a closed subset of A. The chains of a superperfect tree form a copy of the irrationals. If A were Menger, it shouldn't include a closed copy of the space of irrationals, since that space is not Menger.

Theorem 8 (TTT)

If every Menger (Hurewicz) co-analytic subset of $\mathbb{R}$ is σ-compact, then there is an inaccessible cardinal in an inner model.

The proof depends on the well-known (see e.g. [16])

Lemma 9

Assume $\omega_1^{L[a]} = \omega_1$ for some $a \in {}^\omega\omega$. Then ${}^\omega\omega \cap L[a]$ ordered by the relation of $\leq^*$ of eventual dominance has a co-analytic ω_1-scale, i.e. a cofinal subset A, well-ordered by $\leq^*$ in order-type ω_1, which does not include a perfect set (and hence is not σ-compact).

We may make this assumption, else ω_1 is inaccessible in an inner model and we are done. Suppose then that we have such an A. If A is Menger, we're done, so suppose not. Then by Hurewicz 1928, there is a continuous $f : A \to {}^{\omega}\omega$ with range cofinal in $({}^{\omega}\omega, \leq^*)$. By Lavrentieff's Theorem, f extends to a continuous map on a $G_\delta \supseteq A$. Then there is a Borel map $g : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $g \restriction A = f$ (see [17]). Let $b \in {}^{\omega}\omega$ code both a and g. Then ${}^{\omega}\omega \cap L[b]$ is cofinal in $({}^{\omega}\omega, \leq^*)$. Applying the Lemma again, we obtain a co-analytic ω_1-scale B in $({}^{\omega}\omega, \leq^*)$. But Bartoszynski and Tsaban (2006) show a scale $\cup[\omega]^{<\omega}$ is Menger and not σ-compact. A countable set of reals is co-analytic, and the union of two co-analytic sets of reals is co-analytic. □

A minor variation of this proof yields

Theorem 10 (TTT)

If every co-analytic Hurewicz set of reals is σ-compact, then there is an inaccessible in an inner model.

Problem 3

Can Hurewicz' theorem be extended to non-metrizable spaces?

Spaces that are not necessarily metrizable

Definition 5

A space is **analytic** if it is a continuous image of the space $\mathbb{P}$ of irrationals.

Proposition 11 (Arhangel'skiĭ 1986)

Analytic Menger spaces are σ-compact.

Theorem 12 (TT)

Čech-complete Menger spaces are σ-compact.

Proof.

A map is **perfect** if it is continuous, closed, and inverse images of points are compact. A Čech-complete Lindelöf space is a perfect pre-image of a separable metrizable space. A perfect image of a Čech-complete space is Čech-complete. A continuous image of a Menger space is Menger. A perfect pre-image of a σ-compact space is σ-compact. □

Definition 6

A space is **co-analytic** if some remainder of it is analytic.

Problem 4 (Open)

Is it consistent that co-analytic Menger spaces are σ-compact?

Example 1

There is a Menger continuous image of a co-analytic space which is not σ-compact.

Okunev's space Take the Alexandrov duplicate of $\mathbb{P}$ (the space of irrationals) and collapse the non-discrete copy of $\mathbb{P}$ to a point. The Alexandrov duplicate of a space X is defined as follows. Take two disjoint copies X_1 and X_2 of X. The points in X_2 are taken to be isolated; for a point $x \in X_1$, a neighborhood is obtained by taking a neighborhood U of x in X, taken as a subset of X_1, together with the set $U \setminus \{x\} \subseteq X_2$. See Burton-Tall 2012 for details.

Theorem 13 (Tall 2016, Tokgöz 2016)

It is undecidable (modulo an inaccessible) whether co-analytic Menger topological groups are σ-compact.

One direction uses **OGA*(projective)**; the other, $V = L$.

Productive Lindelöfness

Definition 7

A space X is **productively Lindelöf** if for every Lindelöf Y, $X \times Y$ is Lindelöf.

Proposition 14

Productively Lindelöf spaces are consistently Menger (e.g. under CH). (Repovs-Zdomskyy 2012, Alas-Aurichi-Junqueira-Tall 2011, Tall 2013)

Problem 5

Are productively Lindelöf co-analytic spaces σ-compact?

Theorem 15

CH implies productively Lindelöf, nowhere locally compact co-analytic spaces are σ-compact.

Proof.

The analytic remainder is dense in the compactification. Then $w(X) \leq 2^{\aleph_0} = \aleph_1$, so X is Alster and hence (Aurichi 2010) Menger. □

Theorem 16

There is a Michael space (i.e. a Lindelöf space X such that $X \times \mathbb{P}$ is not Lindelöf) iff productively Lindelöf Čech-complete spaces are σ-compact.

What is the right generalization of definability to arbitrary topological spaces?

Definition 8 (Frolík)

A space is **K-analytic** if it is a continuous image of a Lindelöf Čech-complete space.

Example 2

Okunev's space is a K-analytic, productively Lindelöf, Menger space which is not σ-compact. This shows that the following result cannot be strengthened beyond "co-analytic".

Theorem 17

K-analytic co-analytic Menger spaces are σ-compact.

Proof.

Such a space X is a Lindelöf p-space, i.e. a perfect pre-image of a separable metrizable space, since both it and its remainder are Lindelöf $\sum$ (continuous images of Lindelöf p-spaces) (Arhangel'skiĭ 2005). Let X map perfectly onto a metrizable M. Then M is analytic and Menger, so is σ-compact, so X is also. □

Definition 9 (Arhangel'skiĭ 2000)

A space is **projectively σ-compact** if each continuous separable metrizable image of it is σ-compact.

Lemma 18 (Tall 2013)

Projectively σ-compact spaces have finite powers Hurewicz.

Theorem 19

Every K-analytic Menger space is Hurewicz.

Proof.

Each such space is projectively σ-compact. □

Okunev's space is projectively σ-compact and hence Hurewicz.

Definition 10 (Rogers-Jayne 1980)

A space is **K-Lusin** if it is an injective continuous image of $\mathbb{P}$.

Problem 6 (Open)

Is every Menger K-Lusin space σ-compact?

Lemma 20 (Rogers-Jayne 1980)

The following are equivalent for a K-Lusin X:

(a) *X includes a compact perfect set;*

(b) *X admits a continuous real-valued function with uncountable range;*

(c) *X is not the countable union of compact subspaces which include no perfect subsets. In particular, if X is not σ-compact, it includes a compact perfect set.*

From this, we can conclude that Okunev's space is not K-Lusin, since it is not σ-compact but doesn't include a compact perfect set.

Indeed we have:

Definition 11

A space is **Rothberger** if whenever $\{\mathcal{U}_n\}_{n<\omega}$ are open covers, there exists a cover $\{U_n\}_{n<\omega}, U_n \in \mathcal{U}_n$.

Thus *Rothberger* is a strengthening of *Menger*.

Lemma 21 (folklore, Aurichi 2010)

Rothberger spaces do not include a compact perfect set.

Theorem 22

K-analytic Rothberger spaces are projectively countable.

Proof.

They are projectively σ-compact. □

Corollary 23

K-Lusin Rothberger spaces are σ-compact.

Proof.

This follows from Lemma 20. □

A good candidate for definability in general spaces

Theorem 24

The following are equivalent:

(a) *X is proper K-Lusin, i.e. perfect pre-image of an injective image of $\mathbb{P}$ in $\mathbb{R}^\omega$,*

(b) *X and its remainder are K-Lusin,*

(c) *X and its remainder are K-analytic,*

(d) *X and its remainder are both Frolík, i.e. the intersection of countably many σ-compact subspaces in their Čech-Stone compactification, i.e. X is both absolute $F_{\sigma\delta}$ and absolute $G_{\delta\sigma}$,*

(e) *X is Lindelöf Borelian of the first type i.e. generated from the open sets in its Čech-Stone compactification by taking countable intersections and unions,*

(f) X is a Lindelöf p-space which is *absolute Borel*, i.e. generated from the closed subsets of its Čech-Stone compactifications by taking countable unions and intersections,

(g) X is absolute Borel and of *countable type*, i.e. each compact subspace is included in a compact subspace which has a countable neighborhood base for the open sets around it.

Corollary 25

Menger proper K-Lusin spaces are σ-compact.

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