Selection principles around definable algebraic structures

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Introduction and some basic definitions

The oldest well-known selection principles

Menger property

“ property E ” (1924, Menger)

“ property M ” (1988, Miller-Fremlin)

Hurewicz property (1925)

Rothberger property (1938)

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Menger's Conjecture. A metric space X satisfies the Menger property if and only if X is σ-compact.

Menger's conjecture is *false* under CH (Hurewicz),
in ZFC (Miller-Fremlin), (Chaber-Pol), (Tsaban-Zdomskyy)

Notations

$[\mathbb{N}]^\infty$: the family of infinite subsets of $\mathbb{N}$

$\mathbb{N}^{<\infty}$: the family of finite subsets of $\mathbb{N}$

Definition

A **tower** of cardinality κ is a set $T \subseteq [\mathbb{N}]^\infty$ which can be enumerated bijectively as $\{x_\alpha : \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $x_\beta \subseteq^* x_\alpha$.

A set $B \subseteq [\mathbb{N}]^\infty$ is **unbounded** if the set of all increasing enumerations of elements of B is unbounded in $\mathbb{N}^\mathbb{N}$, with respect to $\leq^*$.

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Definition

A subset A of a Polish space X is **co-analytic** if $X \setminus A$ is analytic (i.e., $X \setminus A$ is a continuous image of the space $\mathbb{P}$ of irrationals).

Gödel's universe of constructible sets:

- $L_0 = \emptyset$, $L_{\alpha+1} = def(L_\alpha)$,
- $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ if α is a limit ordinal, and
- $L = \bigcup_{\alpha \in Ord} L_\alpha$.

The Axiom of contructibility state that $V = L$.

Under $V = L$, we can an encoding argument which was first used by Erdös, Kunen and Mauldin (1981). A general method was given by Miller (1989).

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Theorem 1

$V = L$ implies there is a co-analytic unbounded tower.

Proof. Assume $V = L$. By using the well-ordering $<_L$ on L one can construct a Σ^1_2 set of the reals X.
Therefore, there is a co-analytic set $B \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $p(B) = X$ where p is the projection map on the first coordinate.
By Kondô's Uniformization Theorem there exists a co-analytic set $C \subset B$ which is a graph of a function f such that the domain of f is X. Use an arithmetical coding on C and define a co-analytic set of reals C' .

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Theorem (Miller-Fremlin(1988))

$V = L$ implies there is a co-analytic Menger set of reals which is not σ-compact.

But we have a stronger result:
Gerlits and Nagy (1982) introduced a covering property which satisfies all of the former selection principles mentioned above.

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Theorem 2

$V = L$ implies there is an uncountable co-analytic γ-set.

Proof.
Following Theorem 1, there is an unbounded co-analytic tower T of size $\mathfrak{p} = \aleph_1$. Then $T \cup \mathbb{N}^{<\infty}$ satisfies the γ-property (Orenshtein-Tsaban, 2011).
The family of co-analytic sets Π^1_1 contains all Borel sets and is closed under countable unions.

Fact. Uncountable γ-sets never be σ-compact.

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Question

Is the Menger (Hurewicz) conjecture true for co-analytic topological groups?

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By Theorem 2, there is an uncountable co-analytic γ-set. Since γ-property is linearly σ-additive, hereditary for closed subsets and preserved by continuous images, there is a subgroup of reals which satisfies γ-property (O-T).

Recall that a map $f\colon X \to Y$ between two topological spaces X and Y is *Borel (measurable)* if the inverse image of a Borel (equivalently, open or closed) set is Borel.

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Definition(Barr, Kennison, Raphael, 2007)

A topological space is *productively Lindelöf* if its product with every Lindelöf space is Lindelöf.

- (Michael) *CH* implies every productively Lindelöf metrizable space is σ-compact.
- $V = L$ implies *CH*.

Corollary

$V = L$ implies there is a co-analytic γ-subgroup of reals which is not productively Lindelöf.

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Gödel constructibility was generalized by Levy and Shoenfield to relative constructibility which gives a transitive model $L[a]$ of ZFC for any set a.

Theorem

Suppose ${\omega_1}^{L[a]} = \omega_1$ for some $a \in \mathbb{N}^{\mathbb{N}}$. If $\mathfrak{p} > \aleph_1$, then there is a co-analytic γ-subgroup of reals which is not productively Lindelöf.

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Proof. The inner model $L[a]$ has a well-ordering $<_{L[a]}$, and using it produce a co-analytic totally imperfect set of reals T of size $\aleph_1$.

Any set of reals of size $< \mathfrak{p}$ is a γ-set due to Galvin-Miller.

By using a similar argument as in Theorem 3, we can obtain a co-analytic γ-subgroup of reals denoted by G_T. Notice that T is a closed subset of G_T and not productively Lindelöf. Thus, G_T cannot be productively Lindelöf. □

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Even if CH fails we have a model:

Corollary

It is consistent that *CH* fails and there is a co-analytic γ-subgroup of reals which is not productively Lindelöf.

Proof. Start with the constructible universe L, and force **MA** $+ 2^{\aleph_0} = \aleph_2$ via a countable chain condition iteration.

In L, there is a co-analytic tower T of cardinality of $\aleph_1$, and $T \cup \mathbb{N}^{<\infty}$ is a co-analytic γ-set.

MA implies $\mathfrak{p} = \mathfrak{t} = \mathfrak{b} = \mathfrak{c}$ and countable chain condition iterations preserve cardinality.

Since $\mathfrak{p} > \aleph_1$ in the extension, $T \cup \mathbb{N}^{<\infty}$ remains a γ-set. $\square$

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We can also separate the Hurewicz and the Rothberger properties under $V = L$.

Theorem

$V = L$ implies there is a co-analytic Rothberger subgroup of reals which is not Hurewicz.

Proof.

There is a co-analytic unbounded tower S. One can identify elements $x \in [\mathbb{N}]^\infty$ with increasing elements of $\mathbb{N}^\mathbb{N}$ by letting $x(n)$ be the nth element in the increasing enumeration of x. Then S is both dominating (under V=L) and well-ordered by $\leq^*$. Fix $S = \{ s_\alpha : \alpha < \mathfrak{d} \} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ where $\mathbb{N}^{\uparrow\mathbb{N}}$ denotes the collection of all increasing elements of $\mathbb{N}^\mathbb{N}$.

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Proof cont.

For each $\alpha < \mathfrak{d}$ pick $a_\alpha \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that

1. $a_\alpha^c \in \mathbb{N}^{\uparrow\mathbb{N}}$, i.e., the complement of the image of a_α is infinite;
2. $a_\alpha \nleq^* s_\alpha$;
3. $a_\alpha^c \nleq^* s_\alpha$.

Define $A = \{ a_\alpha : \alpha < \mathfrak{d} \}$.
$a \in A$ if and only if $\forall s\, \psi(a, s)$ where ψ states the formula given by (1), (2), and (3). Since ψ is arithmetical, A is co-analytic. Therefore, $A \cup \mathbb{N}^{<\infty}$ is co-analytic. $\square$

Remark

The additive group of $\mathbb{R}$ with the usual topology is Borel, in fact σ-compact. Then it is a co-analytic Hurewicz group of reals. Notice that every closed subset of a Rothberger space is Rothberger. And also, every uncountable closed subset of reals contain a perfect subset by the Cantor-Bendixson result. Therefore, $\mathbb{R}$ cannot be Rothberger, since every Rothberger space is totally imperfect.

But we have also:

Theorem

$V = L$ implies there is a co-analytic totally imperfect Hurewicz subgroup of reals which is not Rothberger.

Proof.
SMZ: the collection of strong measure zero subsets of the reals
non(SMZ): the minimal cardinality for a set of reals which does not have strong measure zero.
Fix a $\mathfrak{b}$-scale $H = \{ s_\alpha : \alpha < \mathfrak{b} \}$. Since non$(SMZ) = \aleph_1 = \mathfrak{b}$, there is a set of reals $Y = \{ y_\alpha : \alpha < \mathfrak{b} \}$ which is not strong measure zero.
Define $H' = \{ s'_\alpha : \alpha < \mathfrak{b} \}$, where $s'_\alpha(n) = 2s_\alpha(n) + y_\alpha(n)$ for all n. Then H' is also strongly unbounded and $\mathfrak{b}$-scale. The mapping $\phi \colon H' \to Y$ defined by $s'(n) \to s'(n) \pmod 2$ for all n is a continuous and surjective map.

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Define $H' = \{ s'_\alpha : \alpha < \mathfrak{b} \}$, where $s'_\alpha(n) = 2s_\alpha(n) + y_\alpha(n)$ for all n. Then H' is also strongly unbounded and $\mathfrak{b}$-scale. The mapping $\phi \colon H' \to Y$ defined by $s'(n) \to s'(n) \pmod 2$ for all n is a continuous and surjective map.

Since $\overline{\mathbb{N}^{\uparrow\mathbb{N}}} = \mathbb{N}^{\uparrow\mathbb{N}} \cup \mathbb{N}^{<\infty}$ and $H' \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$, ϕ can be extended to a surjective continuous mapping $\phi^*\colon H' \cup \mathbb{N}^{<\infty} \to Y \cup \mathbb{N}^{<\infty}$.
$\phi^*(H' \cup \mathbb{N}^{<\infty})$ satisfies Hurewicz property. On the other hand, since the property of having strong measure zero is hereditary and $\phi^*(H') = \phi(H')$ does not have strong measure zero, $\phi^*(H' \cup \mathbb{N}^{<\infty})$ does not have strong measure zero, and then it does not satisfy Rothberger property.
For each $y \in Y$ is defined by the arithmetical formula $\forall n(y(n) = s'(n) \pmod 2)$, and so Y is co-analytic. Thus, $Y \cup \mathbb{N}^{<\infty}$ is co-analytic. □

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We also have:

Corollary

Suppose ${\omega_1}^{L[a]} = \omega_1$ for some $a \in \mathbb{N}^{\mathbb{N}}$. If $\mathfrak{d} > \mathfrak{b} = \aleph_1$, then there is a co-analytic Menger subgroup of reals which is neither Hurewicz nor productively Lindelöf.

Corollary

It is consistent that *CH* fails and there is a co-analytic Menger subgroup of reals which is neither Hurewicz nor productively Lindelöf.

Proof.
Start with the constructible universe L. Take any regular cardinal $\kappa > \aleph_1$ such that $\kappa^{\aleph_0} = \kappa$. Then, in the Cohen extension $L[G]$ via Cohen forcing $\mathbb{C}(\kappa)$, we have $\mathfrak{d} > \mathfrak{b} = \aleph_1$. Also, notice that Cohen forcing preserve the cardinality $\aleph_1$, since forcings with countable chain condition (abbreviated c.c.c.) preserve cardinalities. □

THANK YOU!