Homogeneity of ideals

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Joint work with Adam Kwela.

Homogeneity

The *restriction of the ideal* $\mathcal{I}$ *to* $X \subseteq \bigcup \mathcal{I}$ is given by

$$\mathcal{I}|X = \{A \cap X : \; A \in \mathcal{I}\}.$$

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \cong \mathcal{J}$ if there is a bijection $f : \bigcup \mathcal{I} \longrightarrow \bigcup \mathcal{J}$ such that $f[C] \in \mathcal{J} \iff C \in \mathcal{I}$.

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Theorem

When $A \in H(\mathcal{I})$ and $B \supseteq A$ then $B \in H(\mathcal{I})$.

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We call an ideal $\mathcal{I}$ on ω *homogeneous* if $H(\mathcal{I}) = \mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$.

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All weakly homogeneous ideals are homogeneous.

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- Let $\{I_n : n \in \omega\}$ be a family of consecutive intervals such that each I_n has length $n!$. An ideal $\mathcal{I} = \{A \subseteq \omega : \lim_{n\to\infty} |A \cap I_n|/n! = 0\}$ is anti-homogeneous.

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- Ideal of sets of asymptotic density zero $\mathcal{I}_d = \left\{A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A\cap\{0,1,\dots,n\}|}{n+1} = 0\right\}$ is neither homogeneous nor anti-homogeneous.

Invariant functions

Definition (Balcerzak, Głąb, Swaczyna)

Let $\mathcal{I}$ be an ideal on ω and $f:\omega\to\omega$ be an injection. We say that f is:

- *$\mathcal{I}$-invariant* if $f[A]\in\mathcal{I}$ for all $A\in\mathcal{I}$;
- *bi-$\mathcal{I}$-invariant* if $f[A]\in\mathcal{I}\iff A\in\mathcal{I}$ for all $A\subseteq\omega$.

If $f:\omega\to\omega$ is bi-$\mathcal{I}$-invariant then $f[\omega]\in H(\mathcal{I})$. On the other hand, if $A\in H(\mathcal{I})$ then there is a bi-$\mathcal{I}$-invariant $f:\omega\to\omega$ with $f[\omega]=A$.

Fix points of invariant functions

Theorem

The following are equivalent for any ideal $\mathcal{I}$ on ω:

- *there is an $\mathcal{I}$-invariant injection $f : \omega \to \omega$ with $\mathrm{Fix}(f) \notin \mathcal{I}^\star$ and $f[\omega] \notin \mathcal{I}$;*
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Problem

Characterize the ideals for which there are no $A, B \subseteq \omega$ such that $A \triangle B \notin \mathcal{I}$ and $\mathcal{I}|A \cong \mathcal{I}|B$. Specifically, find a "nice" example of such an ideal.

Ideal convergence

Let $\mathcal{I}$ be an ideal on ω. We say that a real sequence $(x_n)_{n\in\omega}$ is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ we have

$$\{n \in \omega : |x_n - x| > \varepsilon\} \in \mathcal{I}.$$

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Proposition

The following are equivalent for any ideal $\mathcal{I}$ on ω not isomorphic to $\mathit{Fin} \oplus \mathcal{P}(\omega)$:

- *for any sequence $(x_n)_{n\in\omega}$ of reals, $\mathcal{I}$-convergence of $(x_n)_{n\in\omega}$ to some $x \in \mathbb{R}$ implies convergence of $(x_{f(n)})_{n\in\omega}$ to x for some bi-$\mathcal{I}$-invariant injection f;*
- *for every countable family $\{A_n : \ n \in \omega\} \subseteq \mathcal{I}$ there exists such $A \in H(\mathcal{I})$ that $A \cap A_n$ is finite for every $n \in \omega$.*

A homogeneous ideal satisfies the above if and only if it is a weak P-ideal. Moreover, an anti-homogeneous ideal satisfies the above if and only if it is a P-ideal.

Enumeration as isomorphism

Theorem

Let $A \in H(\mathcal{I}_d)$ and $\{a_0, a_1 \ldots\}$ be an increasing enumeration of A. Then the function $f : \omega \to A$ given by $f(n) = a_n$ witnesses that $\mathcal{I}_d | A \cong \mathcal{I}_d$.

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Let $\mathcal{I}$ be such that for every $A \in \mathcal{I}$ and $B \subseteq \omega$ when $|B \cap \{0, \dots, n\}| \leq |A \cap \{0, \dots, n\}|$ for almost all $n \in \omega$ then $B \in \mathcal{I}$. Then $\mathcal{I}$ has the property mentioned above

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Problem

Can any other ideal have this property?