Omission of Intervals and real γ-sets

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Frontiers '17

1. Fréchet–Urysohn real function spaces

Fréchet–Urysohn (FU): $x \in \overline{A} \implies A \ni a_n \longrightarrow x$.

Metric (or first countable) $\implies$ FU.

$X \subseteq \mathbb{R}$; $C(X) = \{\text{cont. } f\colon X \to \mathbb{R}\}$, pointwise convergence top.

$f_n \longrightarrow f \iff f_n(x) \longrightarrow f(x) \quad (\forall x)$.

$C(X) \subseteq \mathbb{R}^X$, Tychonoff product.

$C(X)$ is metrizable $\iff$ X is countable.

Gerlits–Nagy '82, following Arhangelskii: Could $C(X)$ be FU?

2. γ-sets

$X \subseteq \mathbb{R}$.

Cover: Proper ($X \notin$ the cover) open cover of X.

$\Uparrow$

ω-cover: $\forall$ finite $F \subseteq X$, $\exists$ in the cover $U \supseteq F$.

$\Uparrow$

Point-cofinite cover: Every point is covered by almost all sets.

X is a γ-set: Every ω-cover contains a point-cofinite cover.

Gerlits–Nagy '82: $\mathrm{C}(X)$ is FU $\iff$ X is a γ-set.

3. Must there be real γ-sets?

Strong measure zero (Borel): $\forall \epsilon_1, \epsilon_2, \ldots$, $\exists$ a cover by intervals with $\mathrm{diam}(I_1) < \epsilon_1$, $\mathrm{diam}(I_2) < \epsilon_2, \ldots$

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Gerlits–Nagy '82: γ-set $\binom{\Omega}{\Gamma}$ $\implies$ strong measure zero.

Laver '76: Consistently, strong measure zero $\implies$ countable.

(Borel's Conjecture)

$\therefore$ Consistently, $C(X)$ is FU only for countable X.

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$\mathfrak{p}$: Min. cardinality of a centered family with no pseudointersection.

$\aleph_1 \leq \mathfrak{p} \leq \mathfrak{c} := |\mathbb{R}|$.

Consistently, $\aleph_1 < \mathfrak{p}$.

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$$X \longrightarrow \mathrm{P}(\mathbb{N})$$

$$x \longmapsto \{\, n \colon x \in U_n \,\}$$

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∴ Consistently, $\aleph_1 < \mathfrak{p}$, and thus $\exists$ uncountable γ-sets. Trivials.

6. Nontrivial γ-sets (the early days)

Galvin–Miller (Taylor) '84: $\mathfrak{p} = \mathfrak{c} \implies$ nontrivial γ-set.

Fin: The finite subsets of $\mathbb{N}$.

Construction in $[\mathbb{N}]^\infty \cup \text{Fin} = \text{P}(\mathbb{N}) = \{0,1\}^{\mathbb{N}} \subseteq \mathbb{R}$.

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$[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$ via inncreasing enumerations: $a = \{a(1), a(2), \dots\}$.

For $a, b \in [\mathbb{N}]^\infty$, $a \leq^* b$: $a(n) \leq b(n)$ for almost all n.

Unbounded in $[\mathbb{N}]^\infty$: $\leq^*$-unbounded.

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Unbounded $\{ x_\alpha : \alpha < \kappa \} \subseteq [\mathbb{N}]^\infty$, $x_\alpha {}^*\!\supseteq x_\beta \quad (\forall \alpha < \beta)$.

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GM '84: $\mathfrak{p} = \mathfrak{c} \implies \exists$ unb. $\mathfrak{p}$-tower T, $T \cup \mathrm{Fin}$ a γ-set.

Proof considers all potential ω-covers (thus $\mathfrak{p} = \mathfrak{c}$).

7. The γ-Set Problem

JMSS '96, Miller '05, Gruenhage–Szeptycki '05,. . . :

T unbounded $\mathfrak{p}$-tower $\implies$ $T \cup \text{Fin}$ is a γ-set? What if $\mathfrak{p} = \aleph_1$?

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$\mathfrak{b}$: Minimal cardinality of an unbounded ($\leq^*$) set in $[\mathbb{N}]^\infty$.

$\mathfrak{p} \leq \mathfrak{b}$, and $\mathfrak{p} < \mathfrak{b}$ is consistent.

$\exists$ unbounded $\mathfrak{p}$-tower $\iff$ $\mathfrak{p} = \mathfrak{b}$.

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Partial answers:

$\mathfrak{p} = \aleph_1 \implies \mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ (Just–Miller–Scheepers–Szeptycki '96),

even $\mathsf{S}_1(\Gamma, \Gamma)$ (Scheepers '98).

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Existence, assuming Dzamonja–Hrusak–Moore $\diamondsuit(\mathfrak{b})$ (Miller '05) .

Weak γ-set (Gruenhage–Szeptycki '05).

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Uses F. Jordan Fusion + insanely intricate diagonalizations.

Proof not conceptual. No one understands it. :(

Goal: A proof I can explain.

9. Omission of intervals

Metric on $P(\mathbb{N})$: $\operatorname{dist}(a,b) := \frac{1}{\min a \Delta b}$.

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$$\mathrm{P}(\{1,\dots,n\}) \subseteq U \text{ open } \implies \exists k > n, \underbrace{\bigcup_{a\in\mathrm{P}(\{1,\dots,n\})} B(a,\frac{1}{k})}_{\{\,x\in\mathrm{P}(\mathbb{N}):\ x\cap(n,k]=\emptyset\,\}} \subseteq U.$$

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Assume $\mathcal{U}$ ω-covers Fin.

$\forall n\ \exists U \in \mathcal{U}, k > n\colon\ x \cap (n,k) = \emptyset \implies x \in U$.

Galvin–Miller Lemma

Let $\mathcal{U}$ ω-cover Fin. Then $\exists\ a \in [\mathbb{N}]^{\infty}$, $U_1, U_2, \ldots \in \mathcal{U}$:

$$x \cap (a(n), a(n+1)) = \emptyset \implies x \in U_n \qquad (\forall x)$$

10. Getting $\mathsf{S}_1(\Gamma,\Gamma)$, via OMI

Scheepers '98, improving JMSS '96:

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$X_\gamma := \mathrm{Fin} \cup \{\, x_\alpha : \alpha < \gamma \,\}$. $< \mathfrak{b}$ $\implies$ $\exists \left\{\, U^n_{m_n} : n \in \mathbb{N} \,\right\} \in \Gamma(X_\gamma)$.

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Key Lemma (Ts '16, following Galvin–Miller)

Let $\mathcal{U}$ ω-cover Fin $\subseteq X$, $|X| < \mathfrak{p}$, and $Y \subseteq [\mathbb{N}]^\infty$ unbounded.

$\exists\ a \in Y$, $U_1, U_2, \ldots \in \mathcal{U}$ a point-cofinite cover of X:

$$b \setminus \{1, \ldots, n\} \subseteq a \implies b \in U_n, U_{n+1}, \ldots \qquad (\forall n).$$

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Proof: Thin $\mathcal{U}$ to a point-cofinite cover of X; GM Lemma

(thin #2); Take the U_n's with omitted intervals (thin #3).

$n <$ n-th omitted interval.

12. Applying the Key Lemma $\aleph_0$ times

$\mathfrak{p} = \mathfrak{b}$. $X = \mathrm{Fin} \cup \{ x_\alpha : \alpha < \mathfrak{b} \}$.

$X_\gamma := \mathrm{Fin} \cup \{ x_\alpha : \alpha < \gamma \}$.

X_{γ_0} arbitrary. By Key Lemma, $\exists$:

$U_1^{(1)}, U_2^{(1)}, \ldots \in \mathcal{U}$ a point-cofinite cover of X_{γ_0},

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Etc $\ldots$. $\gamma := \sup_n \gamma_n < \mathfrak{b}$. $x_\gamma \setminus \{1, \ldots, g(n)\} \subseteq x_{\gamma_n}$.

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$\alpha \geq \gamma \implies x_\alpha \subseteq^* x_\gamma \implies \forall^* n :\ x_\alpha \setminus \{1, \ldots, g(n)\} \subseteq x_{\gamma_n} \implies$
$x_\alpha \in U_{g(n)}^{(n)}, U_{g(n)+1}^{(n)}, \ldots$.

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$x_\alpha \in U^{(n)}_{g(n)}, U^{(n)}_{g(n)+1}, \ldots$. Remove the first $g(n)$ open sets.

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$U_1^{(1)}, U_2^{(1)}, \dots \in \mathcal{U}$ a point-cofinite cover of X_{γ_0},

$$b \setminus \{1, \dots, n\} \subseteq x_{\gamma_1} \implies b \in U_n^{(1)}, U_{n+1}^{(1)}, \dots.$$

$U_1^{(2)}, U_2^{(2)}, \dots \in \mathcal{U}$ a point-cofinite cover of X_{γ_1}, x_{γ_2},

$$b \setminus \{1, \dots, n\} \subseteq x_{\gamma_2} \implies b \in U_n^{(2)}, U_{n+1}^{(2)}, \dots.$$

Etc ... . $\gamma := \sup_n \gamma_n < \mathfrak{b}$. $x_\gamma \setminus \{1, \dots, g(n)\} \subseteq x_{\gamma_n}$.

$\alpha \geq \gamma \implies x_\alpha \subseteq^* x_\gamma \implies \forall^* n :\ x_\alpha \setminus \{1, \dots, g(n)\} \subseteq x_{\gamma_n} \implies$
$x_\alpha \in U_{g(n)}^{(n)}, U_{g(n)+1}^{(n)}, \dots$. Remove the first $g(n)$ open sets.

Any selection is a point-cofinite cover of $\{ x_\alpha : \alpha \geq \gamma \}$!

13. Covering $X_\gamma = \bigcup_n X_{\gamma_n}$

$$X_{\gamma_1} \subseteq X_{\gamma_2} \subseteq \cdots \subseteq X_\gamma.$$

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$$\begin{aligned} X_\gamma &\longrightarrow \mathbb{N}^\mathbb{N} \\ x &\longmapsto f_x(n) = \min\left\{ k \colon x \in U^{(n)}_k, U^{(n)}_{k+1}, \dots \right\} \end{aligned}$$

$$|X_\gamma| < \mathfrak{b} \implies \{ f_x \colon x \in X_\gamma \} \leq^* h.$$

$\therefore$ $U^{(n)}_{h(n)}$ point-cofinite cover of X_γ, thus of X.

60. Goal reached! THANK YOU