Some properties of the intersection ideal $\mathcal{M} \cap \mathcal{N}$

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Terminology

Throughout this talk we deal with subsets of the Cantor space 2^ω with the standard topology, measure and modulo 2 coordinatewise addition denoted by $+$.

$\mathcal{M}$ — σ ideal of meager subsets of 2^ω.
$\mathcal{N}$ — σ ideal of measure zero subsets of 2^ω
$\mathcal{E}$ — σ ideal generated by F_σ measure zero sets in 2^ω.
$\mathcal{E} \subsetneq \mathcal{M} \cap \mathcal{N}$ — σ ideal of sets that are in $\mathcal{M}$ and in $\mathcal{N}$.

Suppose that I and J are σ-ideals of subsets in 2^ω with $I \subseteq J$.

$$X \in (I, J)^* \quad \text{if} \quad \forall\ A \in I \quad X + A \in J.$$

Diagram

We define

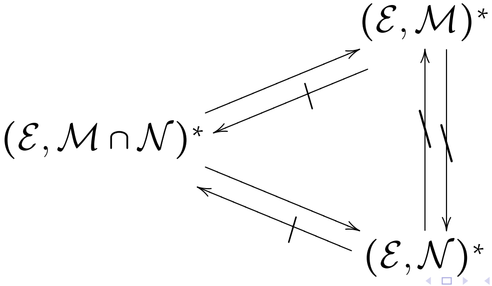
$$I^* = (I, I)^*.$$

1. The following diagram of inclusions holds.

$$\mathcal{N}^* \underset{?}{\rightleftarrows} (\mathcal{M} \cap \mathcal{N})^* \begin{smallmatrix}\longrightarrow \\ \not\longleftarrow\end{smallmatrix} \mathcal{E}^* = \mathcal{M}^* \begin{smallmatrix}\longrightarrow \\ \not\longleftarrow\end{smallmatrix} (\mathcal{E}, \mathcal{M} \cap \mathcal{N})^*$$

Also

2.



Main Theorem

Non-trivial inclusions (or lack of inclusions) in the above diagram:

3. $\mathcal{E}^* = \mathcal{M}^*$ (O. Zindulka). Recall that strongly measure zero sets with Hurewicz property are meager additive (Nowik, Scheepers, W.).
In $M[c]$ there is a set X such that $X \in (\mathcal{E}, \mathcal{M} \cap \mathcal{N})^*$ but $X \notin \mathcal{E}^*$, where c is a Cohen real over M.
Under CH (or $\mathfrak{p} = \mathfrak{c}$) there exists a γ-set $X \notin (\mathcal{M} \cap \mathcal{N})^*$.

$(\mathcal{M} \cap \mathcal{N}, \mathcal{M})^* \to \mathcal{E}^* = \mathcal{M}^*$.

Proof of Main Theorem

We use the latter fact to prove the following Main Theorem.

4. $(\mathcal{M} \cap \mathcal{N})^* \to \mathcal{N}^*$

Proof (sketch). Suppose $X \in (\mathcal{M} \cap \mathcal{N})^*$. Then by previous remark $X \in \mathcal{M}^*$. By Bartoszyński–Judah–Shelah characterization, for every $f \in \omega^{\omega\uparrow}$ there are $g \in \omega^{\omega\uparrow}$ and $y \in 2^\omega$, so that

$$X \subseteq \{x \in 2^\omega : \forall_n^\infty \exists k (g(n) \leq f(k) < f(k{+}1) \leq g(n{+}1), \\ \text{and } x \upharpoonright [f(k), f(k+1)) = y \upharpoonright [f(k), f(k+1)))\}.$$

Let

$$\begin{array}{ccccccccc} a_0 < b_0 < \cdots < & a_n & < & b_n & < \\ & \| & & \| & \\ & g(2n) & & g(2n+1) & \end{array}$$

be a sequence of natural numbers sufficiently fast increasing.
For $n \in \omega$, let $T_n \subseteq 2^{[a_n, a_{n+1})}$ be such that $\mu(T_n) < \frac{1}{2^n}$, and $T_n + \langle \sigma_i, \tau_i \rangle$ are stochastically independent for σ_i-distinct. For each $n \in \omega$

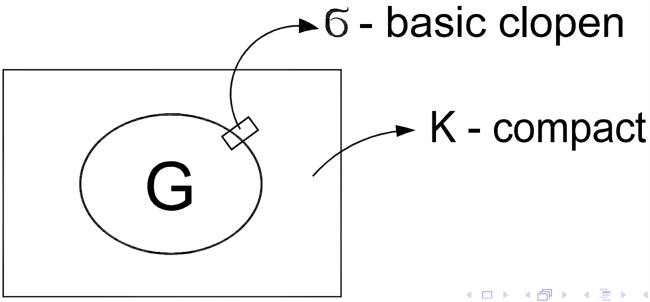
$$A_n = \{x \in 2^\omega : x \upharpoonright [a_n, a_{n+1}) \in T_n\}, \qquad A = \bigcap_{m \in \omega} \bigcup_{n \geq m} A_n.$$

A is of measure zero.

$$\overline{A} = A \cap \{x \in 2^\omega : \forall_n^\infty x \restriction [a_n, a_{n+1}) \neq h \restriction [a_n, a_{n+1})\},$$

for some $h \in 2^\omega$.
Put $\overline{A} = A \cap B$, where B is a meager set in the above formula. Suppose

$$X + \overline{A} \subseteq G, \qquad G\text{-open}, \ \mu(G) < 1.$$



We may assume that for every basic clopen σ as in the picture $\mu(\sigma \setminus G) > 0$.

By de Morgan's law

$$K \subseteq \bigcap_{x \in X} \Big((x + A^c) \cup (x + B^c) \Big) \subseteq \bigcap_{x \in X} (x + A^c) \cup \bigcup_{x \in X} (x + B^c).$$

The union in the expression above is of measure zero. By substracting it we obtain

$$\bigcup_{x \in X} (x + A) \cap \sigma' = \emptyset$$

where every $\sigma' \subseteq \sigma \setminus G$, $\mu(\sigma') > 0$.

Now we can follow the standard procedure to show that there is a sequence

$$\{D_n\}_{n\in\omega}, \qquad D_n \subseteq 2^{[a_n, a_{n+1})}, \qquad |D_n| \leq 2^n,$$

for $n \in \omega$, and such that if $x \in X$, then for almost every n

$$x \restriction [a_n, a_{n+1}) \in D_n.$$

This proves that $X \in \mathcal{N}^*$. □

Open Questions

5. $(\mathcal{M} \cap \mathcal{N}, \mathcal{M})^* \not\to (\mathcal{M} \cap \mathcal{N}, \mathcal{N})^*$. To show this we can use a γ-set $X \notin (\mathcal{M} \cap \mathcal{N})^*$.

6. Question. $(\mathcal{M} \cap \mathcal{N}, \mathcal{N})^* \overset{?}{\to} (\mathcal{M} \cap \mathcal{N}, \mathcal{M})^*$.

7. Question. Is it consistent with ZFC that all members of $(\mathcal{E}, \mathcal{M})^*$ are countable?

We say that $X \in (I, 2^\omega)$ if $\forall A \in I \; X + A \neq 2^\omega$.

8. Question (B. Tsaban). Does ZFC imply that there is an uncountable X in $(\mathcal{E}, 2^\omega)$?

If $\mathfrak{b} = \aleph_1$, then there is an uncountable $X \in \mathcal{M}^*$ (Bartoszyński, Todorcevic).

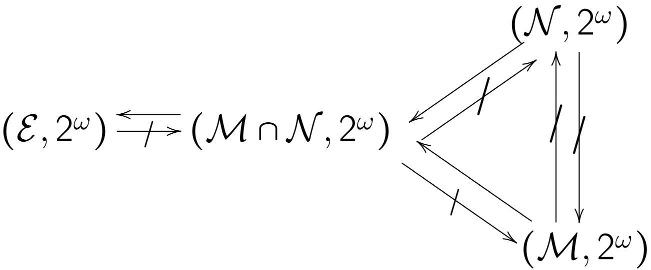
9. If $X \in (\mathcal{E}, 2^\omega)$ and $|X| < \mathfrak{b}$, then $X \in (\mathcal{E}, \mathcal{M})^*$.

10. Positive answer to Question 7 implies negative answer to Question 8 and this in turn proves

$$\text{Con}(\text{ZFC} + \text{Borel conjecture} + \text{dual Borel conjecture})$$

(Goldstern, Keller, Shelah, Wohofsky).

Recall that we say that $X \in (I, 2^\omega)$ if for every $A \in I$, $X + A \neq 2^\omega$.



11. Remark. It is worth noticing that the first crossed arrow holds (under CH) in a more general setting.

12. Question. Is $(\mathcal{M} \cap \mathcal{N}, 2^\omega)$ closed under taking finite unions?

References

- **T. Weiss**, *A note on the intersection ideal* $\mathcal{M} \cap \mathcal{N}$, Comment. Math. Univ. Carolin., 54, 3 (2013).
- **T. Weiss**, *More remarks on the intersection ideal* $\mathcal{M} \cap \mathcal{N}$, Comment. Math. Univ. Carolin., accepted for publication, 2017.
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