Nowhere Differentiable Functions Revisited

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Joint work with Adam Kwela

What is a nowhere differentiable function?

Definition

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Remark

If $n = 1$, *then the definition above collapses to* f *not having a finite derivative at any point of it's domain.*

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- Pathologies are fun!

Classics

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Theorem (Banach 1931)

The set of one-dimensional nowhere differentiable functions defined on a unit interval is residual in $C[0,1]$.

Theorem (Saks 1932)

If we allow infinite derivatives, the set of one-dimensional nowhere differentiable functions defined on a unit interval is meager in $C[0,1]$.

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Modern Mathematics

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- *naively Haar-$\mathcal{I}$ ($A \in \mathcal{NHI}$) if there exists a continuous map $f : 2^\omega \to G$ such that $f^{-1}(A+g) \in \mathcal{I}$ for all $g \in G$.*

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Theorem (Banakh, Jabłońska, Głąb, Swaczyna; $>$ 2017)

The definitions from the previous slide are compatibile.

Theorem (Hunt 1994)

The complement of the set of one-dimensional nowhere differentiable functions defined on a unit interval is Haar-null in $C[0,1]$.

Our results

Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ be nowhere differentiable. If f is even (odd) or it does not have a left-sided (right sided) derivative at any point, then $F(x_1, x_2, ..., x_n) = f\left(\sqrt{{x_1}^2 + {x_2}^2 + ... + {x_n}^2}\right)$ is nowhere differentiable.

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Corollary

There exists a multidimensional nowhere differentiable function.

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Corollary

There exists a multidimensional nowhere differentiable function.

Theorem

The set of multidimensional nowhere differentiable functions defined on a unit cube is residual in $C[0,1]^n$.

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Theorem

The complement of the set of one-dimensional nowhere differentiable functions defined on a unit cube is not Haar-finite in $C[0,1]^n$.

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- What happens with $\mathcal{HN} \cap \mathcal{HM}$ in the multidimensional case?
- Do aliens exist?

Thank you!