Selective versions of separability

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Frontiers of Selection Principles
Cardinal Stefan Wyszynski University, Warsaw
Celebrating the 60th birthday of Marion Scheepers

August 30, 2017

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Combinatorial versions of separability: switching points and open sets

Definition (Scheepers 1999)

X is *M-separable*, if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ so that $\bigcup_{n\in\omega} F_n$ is dense.

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If a countable space X has a π-base of size $< \mathfrak{d}$ (resp. $< \mathfrak{b}$, $< cov(\mathcal{M})$), then it is M-separable (resp. H-separable, R-separable). ○

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Is there a ZFC example of an R-separable space without isolated points which doesn't have a countable π-base? If yes, then what about being not H-separable?

Definition

X has

1. M-*tightness* at $x \in X$ if for every sequence $\langle A_n : n \in \omega \rangle$ of subsets of X s.t. $x \in \bar{A}_n$ for all n, there exists a sequence $\langle B_n : n \in \omega \rangle$ such that $B_n \in [A_n]^{<\omega}$ and any open $O \ni x$ meets some B_n;

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Observation

Suppose that a separable space has M-(resp. H-, R-)tightness ar each point. Then it is M-(resp. H-, R-)separable. ○

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In case of M and H we'll use the following

Definition (Tsaban 2014)

$f : X \to \omega^\omega$ is *upper continuous* if $\{x : f(x)(n) < m\}$ is open for all $n, m \in \omega$.

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Note that

$D_n = \{f : f \upharpoonright (X \setminus U) = 1$ for some $U \in \mathcal{U}_n^1\}$

is dense in $C_p(X)$.

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We'll consider the cases of M and H, R is a bit more subtle. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of X^k. Take $k = 2$. Wlog, each $\mathcal{U}_n$ is closed under finite unions. Passing to refinements, we may assume that all $\mathcal{U}_n$ consist of sets of the form U^2, where $U \subset X$ is open, and $\mathcal{U}_n^1 := \{U : U^2 \in \mathcal{U}_n\}$ is an ω-cover of X for all n. ○

Note that

$D_n = \{f : f \restriction (X \setminus U) = 1$ for some $U \in \mathcal{U}_n^1\}$

is dense in $C_p(X)$. Use M- or H-separability of $C_p(X)$. ○

To close the circle, we need

Lemma

For a metrizable space X, if $C_p(X)$ is

1. *M-separable , then all finite powers of X are Menger;*
2. *R-separable, then all finite powers of X are Rothberger.*
3. *H-separable, then all finite powers of X are Hurewicz.* ∘

We'll consider the cases of M and H, R is a bit more subtle. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of X^k. Take $k = 2$. Wlog, each $\mathcal{U}_n$ is closed under finite unions. Passing to refinements, we may assume that all $\mathcal{U}_n$ consist of sets of the form U^2, where $U \subset X$ is open, and $\mathcal{U}_n^1 := \{U : U^2 \in \mathcal{U}_n\}$ is an ω-cover of X for all n. ∘

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$D_n = \{f : f \upharpoonright (X \setminus U) = 1 \text{ for some } U \in \mathcal{U}_n^1\}$

is dense in $C_p(X)$. Use M- or H-separability of $C_p(X)$. ∘

The last slide

Thank you for your attention.

Happy Birthday, Marion!