$\mathcal{I}$-Luzin sets

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Definition

A family $\mathcal{I}$ is a σ-ideal of subsets of $\mathbb{R}^n$, if it is closed under countable unions and taking subsets.

$\mathcal{B}$ denotes the family of Borel subsets of $\mathbb{R}^n$.

Definition

A σ-ideal $\mathcal{I}$:

- is translation invariant if for each $\bar{x} \in \mathbb{R}^n$ and $A \in \mathcal{I}$
$$\bar{x} + A = \{\bar{x} + \bar{a} : \ \bar{a} \in A\} \in \mathcal{I};$$
- has a Borel base if $(\forall I \in \mathcal{I})(\exists B \in \mathcal{B} \cap \mathcal{I})(I \subseteq B)$.

$$A + B = \{a + b : \ a \in A, b \in B\}, \quad \bigoplus^n A = \underbrace{A + \cdots + A}_{n}.$$

Definition

A set A is

- $\mathcal{I}$-Luzin set, if for each $I \in \mathcal{I}$ we have $|A \cap I| < |A|$.
- super $\mathcal{I}$-Luzin set, if A is an $\mathcal{I}$-Luzin set and for each $\mathcal{I}$-positive Borel set B we have $|A \cap B| = |A|$.

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- L is a generalized Luzin set if L is $\mathcal{M}$-Luzin set
- S is a generalized Sierpiński set if S is $\mathcal{N}$-Luzin set.

$\mathcal{M}$ is σ-ideal of meager sets,
$\mathcal{N}$ is σ-ideal of null sets.

Definition

- $\mathcal{I}$ has the weaker Smital property, if there exists a countable dense set D such that for each $A \in \mathcal{B} \setminus \mathcal{I}$ the set $(A + D)^c \in \mathcal{I}$.

  Bartoszewicz A., Filipczak M., Natkaniec T.,On Smital properties, Topology and its Applications (2011), Vol 158, 2066-2075.

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- The set D witnesses that $\mathcal{I}$ has the weaker Smital property.
- $\mathcal{I}$ has the Smital property if for every dense set D and every positive Borel set $B \notin \mathcal{I}$, the set $(B + D)^c \in \mathcal{I}$.
- $\mathcal{I}$ has the Steinhaus property if for every $A, B \in \mathcal{B} \setminus \mathcal{I}$, the set $A + B$ has nonempty interior.

$$\text{Steinhaus} \implies \text{Smital} \implies \text{weaker Smital}$$

Let $\mathcal{I} \subseteq P(\mathbb{R}^k)$ and $\mathcal{J} \subseteq P(\mathbb{R}^m)$ be σ-ideals. We define a Fubini product $\mathcal{I} \otimes \mathcal{J} \subseteq P(\mathbb{R}^{k+m})$ as follows:

$$A \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow (\exists B \in \mathcal{B})(A \subseteq B \wedge \{\bar{x} \in \mathbb{R}^k : B_{\bar{x}} \notin \mathcal{J}\} \in \mathcal{I})$$

Balcerzak M., Kotlicka E. Steinhaus property for products of ideals, Publ. Math. Debrecen 63 (2003), 235-248.

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Theorem (Bartoszewicz, Filipczak, Natkaniec, 2011)

If $\mathcal{I}_1$ and $\mathcal{I}_2$ have the weaker Smital property then $\mathcal{I}_1 \otimes \mathcal{I}_2$ also has the weaker Smital property.

Lemma

Let P and Q be perfect subsets of $\mathbb{R}^n$. Then there exist perfect sets $P' \subseteq P$ and $Q' \subseteq Q$ such that for each $x \in \mathbb{R}^n$

$$|(P' + x) \cap Q'| \leq 1.$$

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Proof.

Let P' and Q' be perfect subsets from the Lemma $P = Q = \mathbb{R}^n$. Set $\mathcal{J}$ to be a σ-ideal generated by translations of P' i.e.

$$\mathcal{J} = \{X \subseteq \mathbb{R}^n : \ (\exists C \in [\mathbb{R}^n]^\omega)(X \subseteq P' + C)\}.$$

□

Theorem

Assume that $\mathcal{I}$ has the weaker Smital property. Then $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable.

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Proof.

Let L be a Borel $\mathcal{I}$-Luzin.
Find two perfect sets P and Q contained in L, by Lemma for each $x \in X$

$$|(P + x) \cap Q| \leq 1.$$

$P,\ Q \notin \mathcal{I}$. By the weaker Smital property $(P + D)^c \in \mathcal{I}$ and $(P + D) \cap Q \notin \mathcal{I}$.
Clearly $(P + D) \cap Q$ is countable. Contradiction. □

Theorem

Assume that $\mathcal{I}$ has the weaker Smital property. Then the existence of $\mathcal{I}$-Luzin set implies the existence of a super $\mathcal{I}$-Luzin set.

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Proof.

D witnesses that $\mathcal{I}$ has the weaker Smital property. We may assume that $D = D + D = -D$.
Let L be an $\mathcal{I}$-Luzin set such that $cf(|L|) > \omega$.
$L - D$ is a super $\mathcal{I}$-Luzin set. □

Definition

$\mathcal{I}$ is tall, if $\quad \forall B \in \mathcal{B} \setminus ctbl \quad \exists P \in \mathrm{Perf} \cap \mathcal{I} \quad P \subseteq B$

Theorem (Michalski, 2016)

$\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable
iff
$\mathcal{I}$ is tall.

Michalski M., On some relations between properties of invariant σ-ideals in Polish spaces, 14th Students' Science Conference, Fundamental research questions, (2016), 29-33

Lemma

Let L be an $\mathcal{I}$-Luzin set. Then there exists a linearly independent $\mathcal{I}$-Luzin set.

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Assume that $\mathcal{I}$ has the weaker Smital property.
Let L be an $\mathcal{I}$-Luzin set of cardinality $\mathfrak{c}$.
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Question

Does the existence of an $\mathcal{I}$-Luzin set of size $\mathfrak{c}$ imply the existence of an $\mathcal{I}$-Luzin set which is a Hamel base?

Theorem

Let L be a linearly independent $\mathcal{I}$-Luzin set of cardinality $\mathfrak{c}$. Then there exists a set X such that $\{x + L : x \in X\}$ is a partition of $\mathbb{R}^n$.

Theorem (CH)

For each linearly independent $\mathcal{I}$-Luzin set L there exists an $\mathcal{I}$-Luzin set X such that $\{x + L : x \in X\}$ is a partition of $\mathbb{R}^n$.

Theorem (CH)

- Assume that for every $A \in \mathcal{I}$, $\frac{1}{2}A \in \mathcal{I}$. Then there exists an $\mathcal{I}$-Luzin set L such that $L + L$ is an $\mathcal{I}$-Luzin set.
- Assume that for every $A \in \mathcal{I}$, $-A \in \mathcal{I}$. Then there exists an $\mathcal{I}$-Luzin set L such that $L + L = \mathbb{R}^n$.
- Assume that $\mathcal{I}$ is closed under rational scaler multiplication i.e. $(\forall x \in \mathbb{Q})(\forall A \in \mathcal{I})(xA = \{xa : \ a \in A\} \in \mathcal{I})$. Then for each $m \in \omega \setminus \{0\}$ there exists an $\mathcal{I}$-Luzin set L such that $\bigoplus^m L$ is an $\mathcal{I}$-Luzin set and $\bigoplus^{m+1} L = \mathbb{R}^n$.
- Assume that $\mathcal{I}$ is closed under rational scaler multiplication. Then there is a linearly independent (over $\mathbb{Q}$) $\mathcal{I}$-Luzin set L such that $span(L)$ is $\mathcal{I}$-Luzin set.

Corollary (CH)

Assume that $\mathcal{I}$ is scaling invariant.

1. There exists an $\mathcal{I}$-Luzin set L such that $\bigoplus^{m+1} L$ is an $\mathcal{I}$-Luzin for each $m \in \omega$,
2. There exists an $\mathcal{I}$-Luzin set L such that $L + L = L$,
3. There exists an $\mathcal{I}$-Luzin set L such that $\langle \bigoplus^{m+1} L : m \in \omega \rangle$ is a ascending sequence of $\mathcal{I}$-Luzin sets.

Theorem

It is consistent that $\mathfrak{c} = \omega_2$ and there is a Luzin set which is a linear subspace of $\mathbb{R}^n$.

Proof.

Let V' be a model obtained from a model V of CH by adding ω_2 Cohen reals $\{c_\alpha : \alpha < \omega_2\}$. Set

$$L = span_{\mathbb{Q}}(\{c_\alpha : \alpha < \omega_2\}).$$

We claim that L is a Luzin set. □

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Question

$\exists$ Luzin set $\implies \exists$ Luzin set which is a linear subspace of $\mathbb{R}^n$?

Definition

B is a Bernstein set, if $\forall P \in \mathrm{Perf} \;\; P \cap B \neq \emptyset \wedge P \cap B^c \neq \emptyset$

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There exists a Luzin set L such that $L + L$ is a Bernstein set.

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Proof.

For meager sets $M_1,\ M_2$ and perfect set P does there exist $l' \in M_2^c$ such that a set $M_1^c \cap (P - l')$ has cardinality $\mathfrak{c}$?
Extend our universe V via generic extension to V' such that

$$V' \models cov(\mathcal{M}) \geq \omega_2.$$

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We will work in V'. Fix a set $A \subseteq P$ of cardinality ω_1.
For every $a \in A$ a set $\{l :\ a - l \in M_1^c\} = -M_1^c + a$ is comeager.

$$\bigcap_{a \in A} \{l :\ a - l \in M_1^c\} \cap M_2^c \neq \emptyset.$$

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$V' \models \exists l' \in M_2^c\ |M_1^c \cap (P - l')| \geq \omega_1.$ □

Theorem (CH)

There exists a Luzin set L such that $L + L$ is a Bernstein set.

Theorem (CH)

There exists a Sierpiński set S such that $S + S$ is a Bernstein set.

Babinkostova L., Sheepers M. Products and selection principles, Topology Proceedings, Vol. 31 (2007), 431-443.

Theorem (Babinkostova, Scheepers, 2007)

If L is a Luzin set and S is a Sierpiński set then a set $L \times S$ is Menger

So, $L + S$ is Menger.
Menger sets are not Bernstein, thus $L + S$ is not a Bernstein set.

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Scheepers, M., Additive properties of sets of real numbers and an infinite game, Quaestiones Math. 16 (1993), 2, 177-191.

Theorem (Scheepers, 1993)

If $A \in \mathcal{N} \cap s_0$ and S is Sierpiński set then $A + S$ has property s_0.

A set A has the property s_0 if for every perfect set P there exists a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$.

Lemma

For every compact null set P there exist a comeager G_δ set G such that $G + P$ is a null set.

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Theorem

Assume that $\mathfrak{c}$ is a regular cardinal. For every generalized Luzin set L and generalized Sierpiński set S a set $L + S$ has the property s_0.

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Proof.

Wlog $|L|=|S|=\mathfrak{c}$.
Let P be a perfect compact null set. By Lemma let G be a comeager null set such that $G+P$ is null. Let $A=-G$ and $B=(G+P)^c$. Then $P\subseteq(A+B)^c$.

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Let P be a perfect compact null set. By Lemma let G be a comeager null set such that $G + P$ is null. Let $A = -G$ and $B = (G + P)^c$. Then $P \subseteq (A + B)^c$.

$$\begin{aligned} L + S &= ((L \cap A) \cup (L \cap A^c)) + ((S \cap B) \cup (S \cap B^c)) \\ &= ((L \cap A) + (S \cap B)) \cup ((L \cap A) + (S \cap B^c)) \cup \\ &\quad \cup ((L \cap A^c) + (S \cap B)) \cup ((L \cap A^c) + (S \cap B^c)) \end{aligned}$$

$(L + S)^c$ contains some perfect subset of P. □

Theorem

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Proof.

Let us start with a model V satisfying GCH. Let P_α be a finite support iteration of $(\dot{Q}_\beta : \beta < \alpha)$ where $\Vdash_\beta \dot{Q}_\beta = \mathcal{C} \times \mathcal{R}_{\aleph_{\beta+2}}$ for $\beta < \alpha$.

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Now, let us work in $V_{\omega_1} = V^{P_{\omega_1}}[G_{\omega_1}]$. Set

$$S = \bigcup_{\alpha \in \omega_1} R_\alpha \cup \{-c_\alpha\}, \quad L = \bigcup_{\alpha \in \omega_1} \{c_\alpha + x : x \in \mathbb{R}^n \cap V_\alpha\}.$$

□

Thank You for Your Attention!

M. Michalski, S. Żeberski sets, "Some properties of I-Luzin sets" Topology and its Applications, 189 (2015), 122-135.