Strong measure zero and the like via selection principles

Ondřej Zindulka

Czech Technical University Prague

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A little bit of history

Borel 1919: definition

A separable metric space X has **strong measure zero** if for any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals there is a cover $\langle U_n : n \in \omega \rangle$ of X such that $\operatorname{dim} U_n \leqslant \varepsilon_n$.
Borel Conjecture (BC): All strong measure zero sets are countable.

Sierpiński 1928:

Under the Continuum Hypothesis, Borel Conjecture fails.

Besicovitch 1932:

Characterization of **Smz** in terms of Hausdorff measures

Prikry–Galvin–Mycielski–Solovay 1973:

A set $X \subseteq \mathbb{R}$ is **Smz** iff $X + M \neq \mathbb{R}$ for each meager set M.

Laver 1976:

Borel Conjecture is relatively consistent with ZFC.

Galvin–Mycielski–Solovay Theorem

Definition (Prikry sets)

$\textbf{Prikry}(\mathbb{G}) = \{A \subseteq \mathbb{G} : \forall M \in \boldsymbol{\mathcal{M}}(\mathbb{G})\ A + M \neq \mathbb{G}\}$.

Theorem (Prikry 1973)

Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d. Then $\textbf{Prikry}(\mathbb{G}) \subseteq \textbf{Smz}(\mathbb{G})$.

Theorem (Galvin–Mycielski–Solovay 1973)

$\textbf{Prikry}(\mathbb{R}) = \textbf{Smz}(\mathbb{R})$

Theorem (σ-compact case — Kysiak 2000, Fremlin 2008)

$\textbf{Prikry}(\mathbb{G}) = \textbf{Smz}(\mathbb{G})$ *for every σ-compact Polish group $\mathbb{G}$.*

Galvin–Mycielski–Solovay Theorem for non-σ-compact groups?

Continuum Hypothesis

- Borel Conjecture makes everything trivial.
- We will thus work in the "opposite" context of CH.

Definition (CH)

$\mathbb{G}$ is GMS if $\mathbf{Prikry}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$, i.e., if GMS Theorem holds for $\mathbb{G}$.

Going to Mexico

Galvin–Mycielski–Solovay Theorem for non-σ-compact groups?

Definition (anti-GMS set)

A set $A \subseteq \mathbb{G}$ is **anti**-GMS if is it nowhere dense and for every sequence $\langle U_n : n \in \omega \rangle$ of open neighborhods of 0 there are $x_n \in \mathbb{G}$ such that for every $z \in \mathbb{G}$, $z + \bigcup_n x_n + U_n$ is dense in A.

Theorem

(CH) *If a Polish group contains an anti-*GMS *set, then it is not* GMS*.*

Theorem (Hrušák Wohofsky Zindulka 2016)

- $\mathbb{Z}^\omega$ *contains an anti-*GMS *set.*
- (CH) $\mathbb{Z}^\omega$ *is not* GMS*.*

Theorem (Hrušák Zapletal)

- (CH) *Every Polish, not locally compact group with a bi-invariant metric contains an anti-*GMS *set.*
- (CH) *A Polish group with a bi-invariant metric is* GMS *if and only if it is σ-compact.*

Hausdorff Measures

Theorem (Besicovitch 1932)

A metric space X is **Smz** *if and only if $\mathcal{H}^g(X) = 0$ for each gauge g*

Hausdorf Measure

- **Gauge:** nondecreasing right-continuous function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$
- **Hausdorff measure** $\mathcal{H}^g(X)$

Corollary

A metric space X is **Smz** *if and only if $\dim_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$.*

Galvin's Game

Mycielski-Solovay Game

Given a metric space X. Define the game $G(X)$:

- Player I plays a number $\varepsilon_n > 0$
- Player II responds with a set $B_n \subseteq X$ such that $\operatorname{diam} B_n \leqslant \varepsilon_n$

Player II wins if $\bigcup_{n\in\omega} B_n = X$.

Theorem (Galvin 2016)

Suppose K is a σ-compact metric space. A set $X \subseteq K$ is **Smz** *if and only if player I has no winning strategy in $G(X)$.*

Borel's definition revisited

Definition (Two Selection Principles)

- $\mathsf{S}_1(\mathcal{A},\mathcal{B})$: for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers of type $\mathcal{A}$ there is a diagonal sequence of sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega \in \mathcal{B}\}$.
- $\mathsf{S}_1(\{\mathcal{A}_n\},\mathcal{B})$ if for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers such that $\mathcal{U}_n \in \mathcal{A}_n$ there is a diagonal sequence of sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\} \in \mathcal{B}$.

Borel's definition revisited

Definition (Uniform covers)

An open cover $\mathcal{U}$ of X is

- a **uniform cover** ($\mathcal{O}^{\text{unif}}$) if there is $\varepsilon > 0$ such that $\forall x\ \exists U \in \mathcal{U}\ B(x, \varepsilon) \subseteq U$
- a **uniform n-cover** (with a fixed n) ($\mathcal{O}_n^{\text{unif}}$) if there is $\varepsilon > 0$ such that $\forall F \in [X]^n\ \exists U \in \mathcal{U}\ B(F, \varepsilon) \subseteq U$
- a **uniform ω-cover** (Ω^{unif}) if it is a uniform n-cover for all $n \in \omega$.

Theorem (Uniform Selections)

The following are equivalent for a metric space:

- **Smz**
- $\mathsf{S}_1(\mathcal{O}^{\text{unif}}, \mathcal{O})$
- $\mathsf{S}_1(\Omega^{\text{unif}}, \mathcal{O})$
- $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{O})$

Smz summary

Characterizations

- Borel's definition
- Selection principles $\mathsf{S}_1(\mathcal{O}^{\mathsf{unif}}, \mathcal{O})$
- Galvin-Mycielski-Solovay Theorem (in σ-compact Polish groups)
- Hausdorff dimension
- Galvin's game (in σ-compact metric spaces)

Meager-additive sets

Definition

$X \subseteq 2^\omega$ is **$\mathcal{M}$-additive** if $X + M \in \mathcal{M}$ for each $M \in \mathcal{M}$.

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is $\mathcal{M}$-additive if an only if

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$
$$g(n) \leqslant f(k) < g(n+1)\ \&\ x{\restriction}\big[f(k), f(k+1)\big) = y{\restriction}\big[f(k), f(k+1)\big)$$

Lemma (vague)

$X \subseteq 2^\omega$ satisfies Shelah's condition if and only if $\overline{\mathcal{H}}^g(X) = 0$ for every gauge g.

Theorem

$X \subseteq 2^\omega$ is $\mathcal{M}$-additive if and only if $\overline{\dim}_{\mathsf{H}}\, f(X) = 0$ for each uniformly continuous $f : X \to Y$.

Shelah's Condition in Groups: Going to Mexico again

Shelah's Condition in Groups

Lemma

Let $\mathbb{G}$ be a locally compact Polish group with an invariant metric and $X \subseteq \mathbb{G}$. If $\overline{\mathcal{H}}^g(X) = 0$ for every gauge g, then X is $\mathcal{M}$-additive.

Lemma

Let $\mathbb{G}$ be a Polish group with an invariant metric and $X \subseteq \mathbb{G}$. If X is $\mathcal{M}$-additive, then $\overline{\mathcal{H}}^g(X) = 0$ for every gauge g.

Theorem

Let $\mathbb{G}$ be a Polish group with an invariant metric and $X \subseteq \mathbb{G}$. Then X is $\mathcal{M}$-additive if and only if $\overline{\dim}_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$.

Borel–like condition

γ-groupable covers

An open cover $\{U_n\}$ of X is γ-**groupable** ($\mathcal{O}^{\gamma\text{-gp}}$) if there is a partition of ω into consecutive intervals I_k such that

$$\left\{\bigcup_{n\in I_k} U_n\right\} \in \Gamma$$

Theorem

The following are equivalent for a metric space X.

- $\overline{\mathcal{H}}^g(X) = 0$ *for every gauge g (Besicovitch-like)*
- *for any sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive reals there is a γ-groupable cover $\langle U_n : n \in \omega\rangle$ of X such that $\dim U_n \leqslant \varepsilon_n$ (Borel-like)*

Selection principles

Theorem (Uniform Selections)

The following are equivalent for a metric space X.

- $\overline{\mathcal{H}}^g(X) = 0$ *for every gauge g (Besicovitch-like)*
- $\mathsf{S}_1(\mathcal{O}^{\mathsf{unif}}, \mathcal{O}^{\gamma\text{-}\mathsf{gp}})$
- $\mathsf{S}_1(\Omega^{\mathsf{unif}}, \mathcal{O}^{\gamma\text{-}\mathsf{gp}})$
- $\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \mathcal{O}^{\gamma\text{-}\mathsf{gp}})$

Galvin–like Game

Mycielski-Solovay Game

Given a metric space X and a class of covers $\mathcal{C}$. Define the game $G(X, \mathcal{C})$:

- Player I plays a number $\varepsilon_n > 0$
- Player II responds with a set $B_n \subseteq X$ such that $\operatorname{diam} B_n \leqslant \varepsilon_n$

Player II wins if $\{B_n\} \in \mathcal{C}(X)$.

Theorem

Suppose K is a σ-compact metric space. A set $X \subseteq K$ satisfies $\mathsf{S}_1(\mathcal{O}^{\text{unif}}, \mathcal{O}^{\gamma\text{-gp}})$ if and only if player I has no winning strategy in $G(X, \mathcal{O}^{\gamma\text{-gp}})$.

$\mathcal{M}$-additive summary

For a set in a locally compact Polish group with an invariant metric.

Characterizations

- $\mathcal{M}$-additive
- Borel-like definition with $\mathcal{O}^{\gamma\text{-gp}}$
- Selection principle $\mathsf{S}_1(\mathcal{O}^{\mathsf{unif}}, \mathcal{O}^{\gamma\text{-gp}})$
- Finitary Hausdorff dimension (Besicovitch-like)
- Galvin-like game $G(X, \mathcal{O}^{\gamma\text{-gp}})$

Some consequences

Proposition

Within the scope of locally compact Polish groups with an invariant metric:

- $\mathcal{M}$*-additive is an intrinsic property*
- $\mathcal{M}$*-additive sets are preserved by continuous maps between groups*
- $\mathcal{M}$*-additive* $\times$ $\mathcal{M}$*-additive is* $\mathcal{M}$*-additive*
- $\mathcal{M}$*-additive* $\times$ **Smz** *is* **Smz**
- $\mathcal{M}$*-additive sets are universally meager and transitively meager*

Selections

- $\mathsf{S}_1(\mathcal{O}^{\text{unif}}, \mathcal{C})$, $\mathsf{S}_1(\Omega^{\text{unif}}, \mathcal{C})$, $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C})$

- $\widehat{\mathcal{C}} = \{\mathcal{U} \cup \mathcal{V} : \mathcal{U} \in \mathcal{C}, |\mathcal{V}| \leqslant \omega\}$

Proposition

- *For any $\mathcal{C}$,*
$$\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \widehat{\mathcal{C}}) \iff \mathsf{S}_1(\Omega^{\text{unif}}, \mathcal{C})$$
- *If $\mathcal{C}$ is countably thick ($\widehat{\mathcal{C}} = \mathcal{C}$), then*
$$\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C}) \iff \mathsf{S}_1(\Omega^{\text{unif}}, \mathcal{C})$$
- *If $\mathcal{C}$ is "reasonable" (including $\mathcal{O}, \mathcal{O}^{\gamma\text{-gp}}$, but not Ω, Γ), then*
$$\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C}) \iff \mathsf{S}_1(\mathcal{O}^{\text{unif}}, \mathcal{C})$$

Games

Modified Game

Given a metric space X and a class of covers $\mathcal{C}$. Define the game $G^*(X, \mathcal{C})$:

- Player I plays a number $\varepsilon_n > 0$
- Player II responds with $F_n \in [X]^n$

Player II wins if $\{B(F_n, \varepsilon_n\} \in \mathcal{C}(X)$.

Theorem

Suppose X is a set in a compact space.

- *X satisfies $\mathsf{S}_1(\mathcal{O}^{\text{unif}}, \mathcal{C})$ if and only if player I has no winning strategy in $G(X, \mathcal{C})$.*
- *X satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C})$ if and only if player I has no winning strategy in $G^*(X, \mathcal{C})$.*

Theorem

Suppose X is a set in a σ-compact space and $\mathcal{C}$ is "reasonable" (including $\mathcal{O}, \mathcal{O}^{\gamma\text{-gp}}, \Omega, \Gamma, \widehat{\Gamma}$)

- *X satisfies $\mathsf{S}_1(\mathcal{O}^{\text{unif}}, \mathcal{C})$ if and only if player I has no winning strategy in $G(X, \mathcal{C})$.*
- *X satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C})$ if and only if player I has no winning strategy in $G^*(X, \mathcal{C})$.*

$\mathcal{N}$-additive sets

Definition ($\mathcal{N}$-additive sets)

A set $X \subseteq 2^\omega$ is **$\mathcal{N}$-additive** if $X + N \in \mathcal{N}$ for all $N \in \mathcal{N}$.

Theorem (Shelah 1995)

A set $X \subseteq 2^\omega$ is $\mathcal{N}$-additive if and only if $\forall f \in \omega^{\uparrow\omega}\ \exists\langle H_n : n \in \omega : n \in \omega\rangle$

- $\forall n \in \omega\ H_n \subseteq 2^{[f(n), f(n+1))}$,
- $\forall n \in \omega\ |H_n| \leqslant n$,
- $X \subseteq \{x \in 2^\omega : \forall^\infty n \in \omega\ x{\restriction}[f(n), f(n+1)) \in H_n\}$.

$\mathcal{N}$-additive sets

Theorem

The following are equivalent for a set $X \subseteq 2^\omega$.

- X *is* $\mathcal{N}$*-additive*
- $\dim_{\mathsf{P}} f(X) = 0$ *for each uniformly continuous* $f : X \to Y$.
- X *satisfies* $\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \Gamma)$
- *Player I has no winning strategy in* $G^*(X, \Gamma)$

Theorem (Follows easily from Tom Weiss 2014)

The same holds for $X \subseteq \mathbb{R}$.

Theorem

Let X be a subset of a σ-compact Polish group with an invariant metric. If X satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \Gamma)$, then it is $\mathcal{N}$-additive.

(T') and perfect measure zero

p.m.z. sets

Repický 1997 in study of "trigonometric thin sets" introduced **perfectly measure zero sets** in $\mathbb{R}$ (combinatorial definition)

(T') sets

Nowik and Weiss 2002 in study of Ramsey Null Sets introduced (T')**-sets** in 2^ω (combinatorial definition)

Theorem

The following are equivalent.

- $X \subseteq 2^\omega$ *is* (T') *($X \subseteq \mathbb{R}$ is perfectly measure zero)*
- $\underline{\dim}_{\mathsf{P}} f(X) = 0$ *for each uniformly continuous* $f : X \to Y$*.*
- X *satisfies* $\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \widehat{\Gamma})$
- X *satisfies* $\mathsf{S}_1(\Omega^{\mathsf{unif}}, \Gamma)$
- *Player I has no winning strategy in* $G^*(X, \widehat{\Gamma})$

Scheepers' Questions

Scheepers paper 1999

Scheepers studied when finite powers of a metric space are **Smz**.
Say that X is **Smz**$^{<\omega}$ if X^n is **Smz** for all $n \in \omega$.

Quote — Scheepers 1999

It would seem that there ought be a characterization [...] in terms of a selection principle of the form $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$. [...] **What $\mathcal{A}$ ought to be eludes me.**

Theorem (My birthday present for Marion)

For a separable metric space X, the following are equivalent.

- X *is* **Smz**$^{<\omega}$
- X *satisfies* $\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \Omega)$
- X *satisfies* $\mathsf{S}_1(\Omega^{\mathsf{unif}}, \Omega)$
- *Player I has no winning strategy in* $G^*(X, \Omega)$

Summary

	Selection Principle	Game	Fractal Dimension	Corresponding Topological Property
Smz	$\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \mathcal{O})$	$G^*(X, \mathcal{O})$	$\dim_{\mathsf{H}}$	Rothberger
Smz$^{<\omega}$	$\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \Omega)$	$G^*(X, \Omega)$	$\dim_{\pi\mathsf{H}}$	Sakai
$\mathcal{M}$-additive	$\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \mathcal{O}^{\gamma\text{-gp}})$	$G^*(X, \mathcal{O}^{\gamma\text{-gp}})$	$\overline{\dim}_{\mathsf{H}}$	Gerlitz–Nagy
(T'), p.m.z.	$\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \widehat{\Gamma})$	$G^*(X, \widehat{\Gamma})$	$\underline{\dim}_{\mathsf{P}}$	γ-set
$\mathcal{N}$-additive	$\mathsf{S}_1(\{\mathcal{O}_n^{\mathsf{unif}}\}, \Gamma)$	$G^*(X, \Gamma)$	$\dim_{\mathsf{P}}$	strong γ-set

A few bits

Definition

Topological variations Say that X **topologically satisfies** $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C})$ if (X, d) satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{C})$ for every metric compatible with X.

Topological variations

- X topologically satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \mathcal{O}) \Leftrightarrow \mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (Rothberger)
- X topologically satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \Gamma) \not\Leftrightarrow \mathsf{S}_1(\{\mathcal{O}_n\}, \Gamma)$

Theorem

The following are equivalent for a zero-dimensional space.

- *X topologically satisfies $\mathsf{S}_1(\{\mathcal{O}_n^{\text{unif}}\}, \Gamma)$*
- *X has the Hurewicz property and is a strong γ^{fin}-set*
- *every continuous image of X in ω^ω is covered by a slalom*

$\mathsf{S}_{\mathrm{fin}}(\mathcal{A}^{\mathrm{unif}}, \mathcal{B})$

$\mathsf{S}_{\mathrm{fin}}(\mathcal{A}^{\mathrm{unif}}, \mathcal{B})$

- $\mathsf{S}_{\mathrm{fin}}(\{\mathcal{O}_n^{\mathrm{unif}}\}, \mathcal{O})$: Menger-bounded, or Small Balls Property
- $\mathsf{S}_{\mathrm{fin}}(\{\mathcal{O}_n^{\mathrm{unif}}\}, \mathcal{O}^{\gamma\text{-gp}})$: σ-totally bounded

Bohuslav Balcar